Certifying Maximal Stretch Ratios

Joseph Hummel, Kaius Noble, Prajith Velicheti

Grad Mentor: Aaron Benda
Faculty Mentor: Jeffrey Danciger

The University of Texas at Austin

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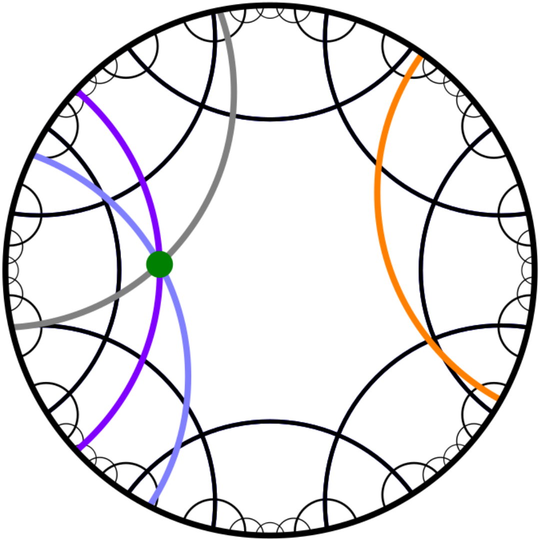
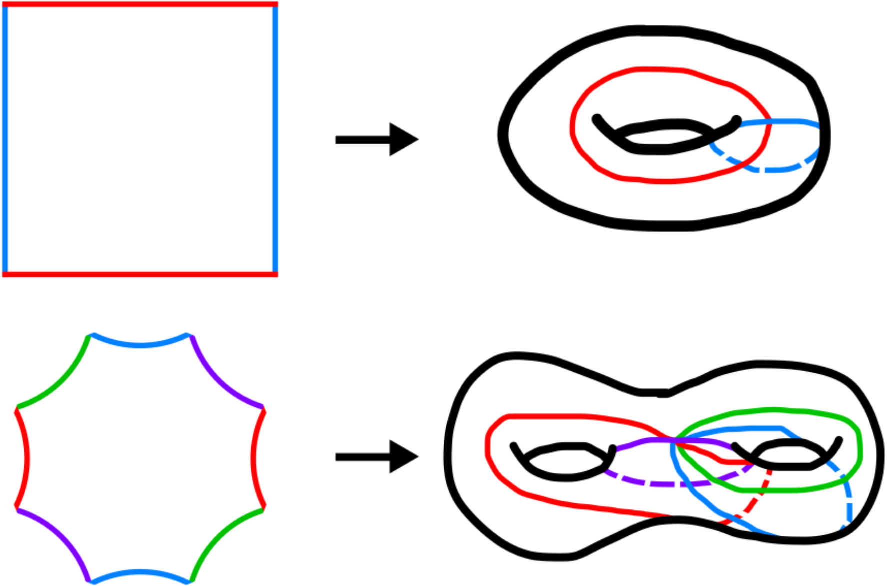
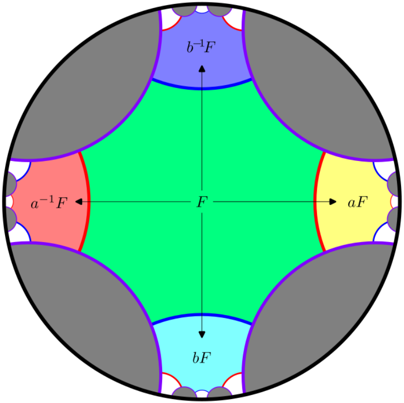
Hyperbolic Geometry

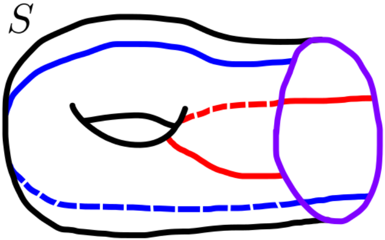


A Hyperbolic Structure on a Surface

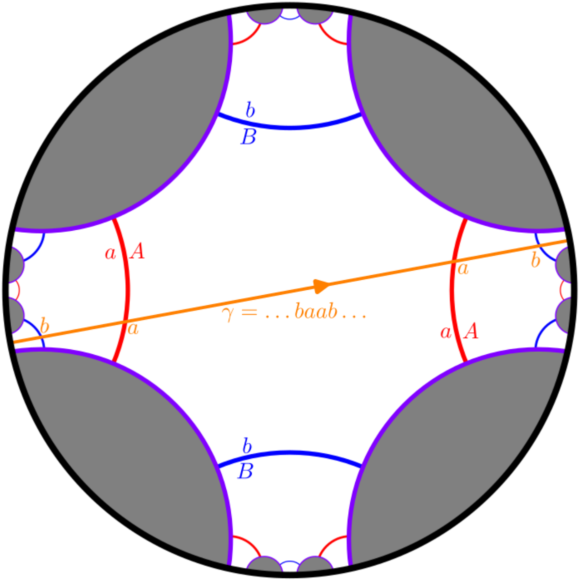


A Hyperbolic Structure on a Surface

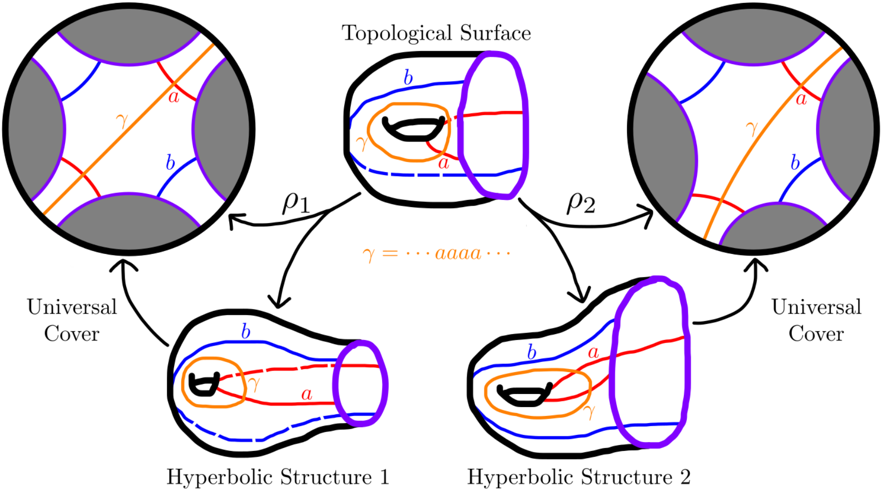




Wall-Letter Association



Comparing Hyperbolic Structures on a Surface



Surfaces and Thurston's Metric

Let $\ell_i(\gamma)$ be the length of the unique geodesic representative in S with respect to some representation ρ_i.

Definition (Thurston's Asymmetric Metric)

$$d_{\mathrm{Th}}(\rho_1, \rho_2) = \log\left(\sup_{\gamma \in \pi_1(S)} \frac{\ell_2(\gamma)}{\ell_1(\gamma)}\right)$$

.

With probability 1, this sup is a max. Our problem is to certify for which geodesic this maximum is obtained.

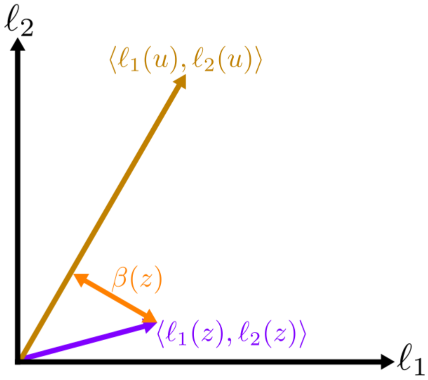
Comparing Geodesics

For the rest of the presentation, fix a guess optimizer $u \in \pi_1(S)$.

Definition ($\beta(z)$)

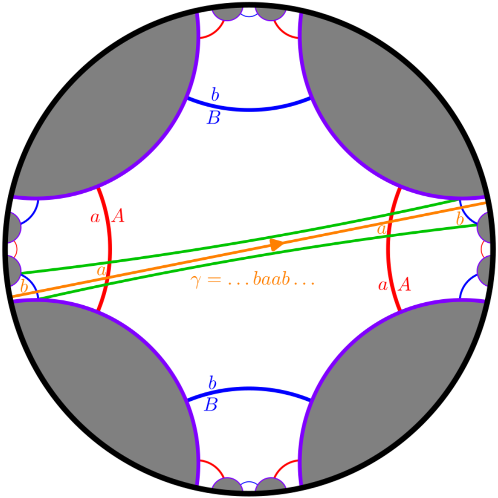
For a word $z \in \pi_1(S)$, we define:

$$\beta(z) := \begin{bmatrix} \ell_1(z) \\ \ell_2(z) \end{bmatrix} \cdot \begin{bmatrix} -\ell_2(u) \\ \ell_1(u) \end{bmatrix}.$$



Preprocessed Words

We define a preprocessed word p to be a reduced word of even length n.



Consider all words of the form:

$$\gamma = \dots p \dots$$

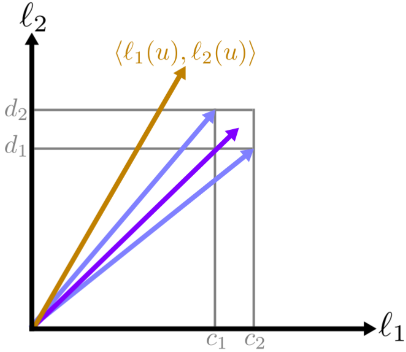
We define $L_i(p)$ to be set of possible lengths.

Comparing Preprocessed Words

Definition ($\beta^P(p)$)

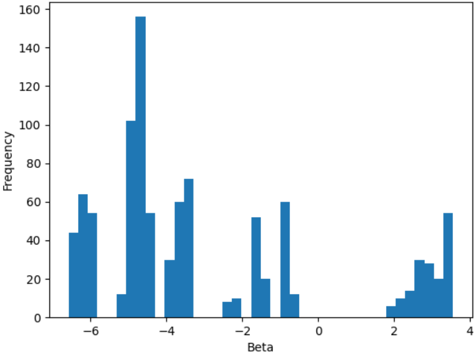
For a preprocessed word p, we define:

$$\beta^P(p) := \begin{bmatrix} \inf(L_1(p)) \\ \sup(L_2(p)) \end{bmatrix} \cdot \begin{bmatrix} -\ell_2(u) \\ \ell_1(u) \end{bmatrix}$$



Progress Check

For a particular example, the distribution of $\beta^P(p)$ for $n = 6$ is below with optimizer ab.



We find $162/972 \sim 16.67\%$ of the preprocessed word betas are positive.

Groupings

To get information of a geodesic over multiple domains, we introduce groupings.

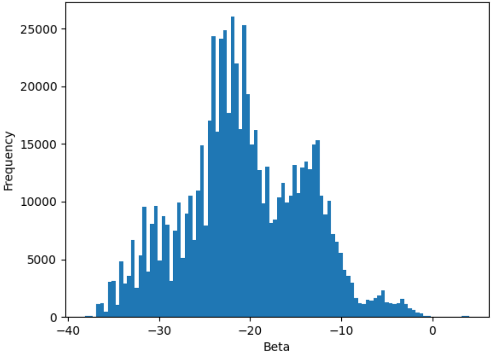
Preprocessed Word (n=8)	Grouping (m=6)
$aBa\underline{aa}BAb$	$aBa\underline{aaBAbba}abb$
Singular Fundamental Domain of Interest	Multiple Fundamental Domains of Interest

Similarly to preprocessed words, we can define:

$$\beta^G(g) := \sum_{p \in g} \beta^P(p).$$

Grouping Distribution

The distribution of groupings for $m = 7$ and $n = 6$ is below.



Of the 708588 groupings, only 162 or 0.023% are positive.

Idea Behind Certification

m	Most Positive Grouping Beta	Beta
2	AB**aba**BA	1.365
4	AB**ababa**BA	1.683
6	AB**abababa**BA	2.000
7	BA**babababa**BA	4.089

Certification

Theorem (Certification)

The algorithm we have informally outlined can certify that a particular word is the optimizer for free group representations.

Converse

Theorem (Converse)

Suppose there exists some $\delta > 0$ such that for every word w that is not the optimizer, we have $\beta(w) \leq -\delta$. Then the algorithm we have outlined will certify the optimizer.