Number Theory Riddles

March 4, 2023

Modular arithmetic

1. Today is Sunday. What day of the week will it be in 13 days?
2. If today were Thursday, what day of the week would it be in 97 days?
3. January 1st, 2023 was a Sunday. What day of the week will it be on January 1st, 2024?
4. Knowing that 2024 is a leap year, what day of the week will it be on January 1st, 2025?

When we think about the days of the week, we are really doing something known as **modular arithmetic**. Let Sunday = 0, Monday = 1, Tuesday = 2, Wednesday = 3, Thurday = 4, Friday = 5, and Saturday = 6. If we count through the days of the week, after Saturday we get back to Sunday, and go to 0 again. If we give each day a number, the day of the week is what is known as the residue class modulo 7. We write $0 \equiv 7 \pmod 7$, which is said as "0 is congruent to 7 modulo 7." This means that a week after Sunday it is Sunday again.

We do the same thing when it comes to telling time (the hours are mod 12, the minutes and seconds mod 60), recognizing if something is even or odd (mod 2), or by recognizing the units digit of a number (mod 10).

Question 1 above has been translated below to modular arithmetic. Can you rewrite the rest of the riddles in the languge of modular arithmetic?

1. What is $0 + 13 \pmod 7$?
2.
3.
4.

Here are some riddles that can be figured out using modular arithmetic:

1. What are the tens and units digits of 7^{2023}? (In other words, can you simplify $7^{2023} \pmod{100}$?)
2. What is the last digit of $(...((7)^7)^7)...)^7$ if there are 1000 7s as exponents and only one 7 in the middle?
3. Can you find a number that is both a multiple of 2 but not a multiple of 4 and a perfect square?

Collatz conjecture

Let's play around with a (deceivingly) simple game of numbers. Consider an arbitrary positive integer n. We will perform the following operation on n:

- If n is even, divide it by two.
- If n is odd, triple it and add one.

We can write this as a function using modular arithmetic:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Now we can form a sequence of integers by repeatedly applying f to a starting number. For instance, begin with 13. We get the following sequence:

$$\begin{gathered} 13 \\ f(13) = 3 \cdot 13 + 1 = 40 \\ f(40) = 40/2 = 20 \\ f(20) = 20/2 = 10 \ldots \end{gathered}$$

1. What happens if you start the sequence at 1?
2. Continue the sequence 13, 40, 20, 10, ... Do you eventually get to 1?
3. What sequence do you get when you start at 53?
4. What sequence do you get when you start at 21?
5. What happens when your sequence reaches a power of 2?
6. What happens when you sequence reaches a number you already know goes to 1?

Sums of squares

1. We can write the prime number 5 as the sum of two squares: $5 = 1^2 + 2^2$. Can you write the following odd prime numbers as a sum of two squares?

   13 29 37 41 61 89 101

2. Is it possible to write 7, 11, or 19 as the sum of two squares?
3. Can you notice anything about the odd primes that you can write as a sum of two squares versus the ones you can't? (Hint: think about modular arithmetic!)
4. Find one more example of a prime that you can write as a sum of two squares, and one which you cannot.

Siegel-Walfisz theorem

Here's a list of all the prime numbers less than 100:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

1. Try sorting these into residue classes mod 3. Let's count how many of them are congruent to 1 (mod 3) and 2 (mod 3), and compare those numbers.

2. What do you think would happen if we counted more primes? Would the numbers begin to even out, or would one residue class consistently have more than the other?

3. Try the same thing going into residue classes mod 5. Now let's count how many are congruent to $1, 2, 3$, and 4 (mod 5).

4. Are you noticing a pattern? Make a guess about what happens in general if we look at how the primes spread out in residue classes, particularly as we count higher and higher.

Goldbach conjecture

Primes are usually thought of as being multiplicative objects, but just as with any integers, we can try adding them. Some interesting things start to happen when you do.

1. Try writing all of the even numbers up to 20 as a sum of two primes (e.g. $6 = 3+3$, $8 = 5+3$). Do any values fail to be written as a sum of two prime numbers?

2. Some even numbers can be written as a sum of two primes in many ways. For example, we can write $94 = 3+91 = 5+89 = 11+83 = 17+77 = 23+71 = 29+65 = 41+53 = 47+47$. That is eight different ways! Do you expect that the larger the number, the more ways we can write it as a sum of primes?

3. How about odd numbers? Can you find any odd numbers that can't be written as a sum of three prime numbers?

4. Do you think any integer can be written as a sum of at most three prime numbers?

Miscellaneous riddles

These riddles are thanks to Cas Monroe!

1. Let p and q be consecutive prime numbers. Can $p+q$ be twice a prime? Prove this.

2. Consider the set of pairs of distinct integers A and B, such that the set of prime factors of A is the same as the set of prime factors of B, and such that the sets of prime factors of $A-1$ and $B-1$ are likewise equal. Is this set of pairs finite or not?

3. Can you find all pairs of prime numbers (p, q) such that $p^q + q^p$ is also a prime number? How do you know that you've found all of them?