Non-Euclidean Geometry

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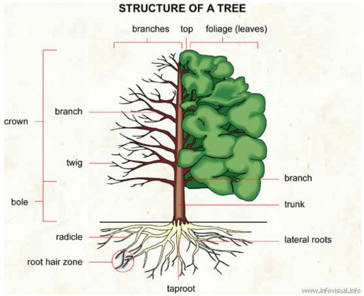
The Axiomatic Method

- The axiomatic method is a method for developing mathematical theories or systems in a concrete and rigorous manner
- In the method, certain axioms are taken to be true
    - These make up fundamental facts about the theory
- Axioms are rules for the system
- **In the axioms, undefined terms are referenced**
    - Without undefined terms, there would be an infinite cycle demanding definitions for terms used in the axioms, and then definitions for terms in the definitions and so on
- **From the axioms alone, theorems are proven**
- No axioms are provable from the other axioms
- The axioms and proven theorems together make up the mathematical system/theory



The History of Euclidean Geometry

- Euclidean geometry is the formalization and systematization of prior knowledge in geometry into rigorous mathematics, through the use of the axiomatic method
- Proved results that had long since been used practically
- The 5 axioms were taken because they were intuitively appealing to the real world
- Euclid was the first to fit all of these propositions into a comprehensive deductive and logical math system
- **For more than 2,000 years, this was the only conceived type of consistent geometry**
- This is a type of synthetic geometry (as opposed to analytic), meaning no coordinates were used; only axioms to propositions
- Results were seen as absolute truths, because the axioms (with the possible exception of the 5th), were so intuitively obvious

Neutral (Absolute) Geometry

- Neutral geometry is geometry in which only Euclid's first 4 axioms and his 5 common notions are taken to be true
- Any propositions or theorems of neutral geometry will hold true in both Euclidean and non-Euclidean geometries (if they take the first four axioms)
  - So its theorems hold in hyperbolic and Euclidean geometry, but not elliptic geometry
- Neutral geometry is an extension of ordered geometry
- **Has an incomplete axiomatic system**
  - This means that other independent axioms can be added to the system without making it **internally inconsistent**
- Euclid's 31st proposition is a theorem of neutral geometry
  - Interesting: the construction of a parallel line to a line through a point not on the line is possible in neutral geometry

The Uncertainty of the 5th Postulate

- The uncertainty surrounding Euclid's 5th postulate - the parallel postulate - was its lack of elegance
- Seemed far less obvious than the others
- Not nearly as simple or straightforward as the others
- Even Euclid clearly did not like it, proving his first 28 propositions entirely in neutral geometry (making no reference whatsoever to the 5th postulate)
- For an extremely long time, many people tried to prove the 5th postulate from the other 4, but could not succeed (we know today that this is impossible)
    - This would mean that neutral and Euclidean geometry were exactly the same thing, and that it is the only conceivable internally consistent geometry

Tried and Failed to Prove the Parallel Postulate

- Ptolemy
- Proclus
- al-Gauhary
- Ibn Al-Haytham
- Omar Kayyám
- Nasir al-Din al-Tusi
- Giordano Vitale
- Johann Lambert
- Saccheri

Failing Methods

- Typically, the false proofs subtly relied on statements that are logically equivalent to the parallel postulate:
  - There is at most one line that can be drawn parallel to another given one through an external point. (Playfair's axiom)
  - The sum of the angles in every triangle is 180° (triangle postulate).
  - **There exists a triangle whose angles add up to 180°.**
  - **The sum of the angles is the same for every triangle.**
  - **There exists a pair of similar, but not congruent, triangles.**
  - Every triangle can be circumscribed.
  - If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.
  - There exists a quadrilateral in which all angles are right angles, that is, a rectangle.
  - There exists a pair of straight lines that are at constant distance from each other.
  - Two lines that are parallel to the same line are also parallel to each other.
  - **In a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides** (Pythagoras' Theorem).
  - **There is no upper limit to the area of a triangle.** (Wallis axiom)
  - The summit angles of the Saccheri quadrilateral are 90°.
  - If a line intersects one of two parallel lines, both of which are coplanar with the original line, then it also intersects the other. (Proclus' axiom)
- Formulating alternative but similar sounding axioms was attempted, but only led to the same logical consequences as the 5th postulate

Ptolemy (90-168)

- Not much is known about him
  - Lived in Alexandria
  - Historians assume that he held Roman Citizenship but was ethnically Greek
- Attempted to prove 5th postulate but subtly relied on what we know today to be playfair's postulate
  - In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

Proclus (410-485)

- Born to a family of nobles in Constantinople
- Began by studying law, drawing his attention towards philosophy
- In Alexandria he began to study the philosophies of Aristotle
- Moved to Athens to study at Plato’s Academy (founded 800 years earlier)
  - Became star student
- Attempted to prove parallel postulate
  - Used the idea that “parallel lines are always a bounded distance apart”
  - However, this statement is logically equivalent to Euclid’s 5th postulate

al-Gauhary (9th Century)

- If alternating angles determined by a line cutting two other lines are equal, then this will hold for all lines cutting the given two.
- Through any point interior to an angle, it is possible to draw a line that intersects both sides of the angle. al-Gauhary makes a hidden assumption: if the alternating angles determined by a line cutting two lines are equal, then the same will be true for all lines cutting the given two

Ibn Al-Haytham (965-1039)

- Arab Muslim scientist, mathematician, astronomer, and philosopher
- Born in Basra, in the Buyid Emirate
- Proposed a hydraulic plan to help improve Nile flooding, but quickly found that the project was impractical, and feigned madness in order to escape punishment
- Attempted to use what we call “Lambert Quadrilaterals” to prove by contradiction the parallel postulate - (Similar to **Nasir al-Din al-Tusi**)
    - Creates a shape with three right angles and a fourth acute angle, which under Euclidean geometry is impossible, but not under Hyperbolic geometry

Omar Kayyám (1050–1123)

- Rejected Euclid’s Postulates in favor of his own
  - Magnitudes are divisible ad infinitum, and are not composed of indivisible things. (Logically equivalent to Euclid’s 5th common notion)
  - One can produce a straight line to infinity. (Logically equivalent to Euclid’s 2nd postulate)
  - Whenever two straight lines intersect, they will diverge when going away from the angle of intersection.
  - **Two converging straight lines will intersect, and two converging lines cannot diverge while going toward convergence. (Logically equivalent to Euclid’s 5th postulate)**
  - Whenever there be two unequal finite magnitudes, the smaller can be multiplied until it becomes greater than the greater.

Saccheri (1667-1733)

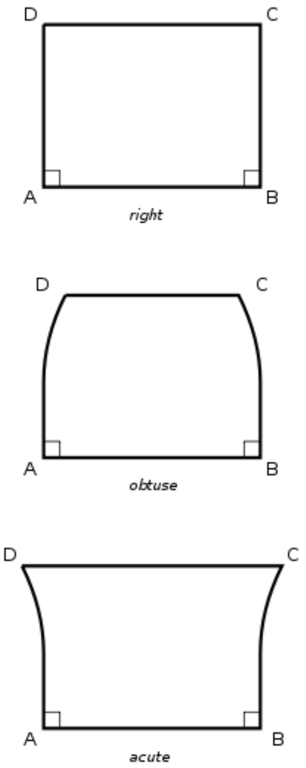
- Saccheri was yet another one of the mathematicians who attempted to prove the parallel postulate from the other four axioms and the five common notions
- He attempted to do this through two proofs by contradiction - assuming that the parallel postulate is not true
    - First Case: Assume $\sum$ angles in a triangle $> 180°$
        - (logically equivalent to there being no parallel lines through a point not on the line)
    - Second Case: Assume $\sum$ angles in a triangle $< 180°$
        - (logically equivalent to there being more than one parallel line through a point not on the line)
- Saccheri believed he had found contradictions in both cases

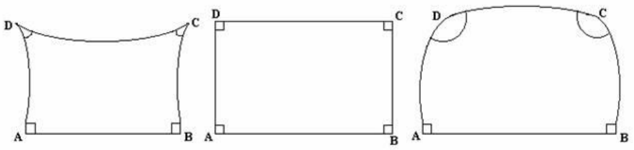
Saccheri’s Contradictions

- First Case: $\sum$ angles in a triangle > 180°
  - Arrived at a contradiction with Euclid’s second postulate, by showing that in this case all lines must be finite
- Second Case: $\sum$ angles in a triangle < 180°
  - The results he was able to derive were so counterintuitive (though not logically contradictory) that he simply concluded they must be false
    - Ex: Theorem: Triangles have a maximum finite area
    - Ex: Theorem: There is an absolute unit of length
  - “The hypothesis of the acute angle is absolutely false; because it is repugnant to the nature of straight lines.” - Saccheri

Saccheri Quadrilaterals

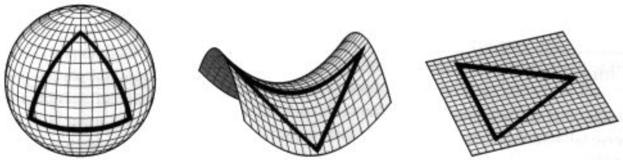
- Used in many of Saccheri's proofs about geometry.
  - Logically equivalent to triangle angles assumptions
  - Used in proofs by Giordano Vitale to make advancements on the understanding of the 5th postulate
- Four sided shape such that the base angles are 90 degrees, but the summit angles vary.
  - ∑ summit angles > 180°
    - Elliptical Geometry
  - ∑ summit angles = 180°
    - Euclidean Geometry
  - ∑ summit angles < 180°
    - Hyperbolic Geometry

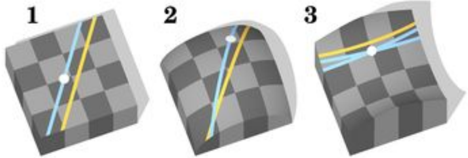




What Saccheri Really Did

- First Case: ∑ angles in a triangle > 180°
    - Provided the basis for elliptical geometry
- Second Case: ∑ angles in a triangle < 180°
    - Provided the basis for hyperbolic geometry
- While Saccheri did not correctly interpret his results to reach these conclusions and discover new geometries, his thinking prompted their discoveries
- If he had directly discovered these geometries, he would've likely become one of the most famous mathematicians ever





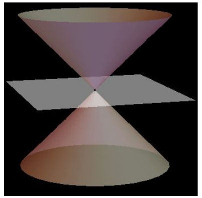
How can you prove that you can't prove the 5th postulate from the other four?

What is a Model?

- A model is a representation of a set of rules (demonstrates the underlying math being done in a situation)
- A model of a consistent theory has all of the properties described by the theory's axioms and theorems
- If there is a model of a theory, that theory is necessarily consistent
    - All that this means is that you can't have a model of an inconsistent theory
        - This should be obvious, based on the fact that an inconsistent theory has contradictions
- If you can make a model that incorporates the first four axioms but not the fifth, you have proven that the fifth is independent and thus cannot be proven from the others

The Inverse Relationship Between the Number of Axioms and the Number of Models

- Theory consistent ⇔ There exists a model
- The more axioms that there are, the fewer models can be used to describe the system
  - Pool party example (what rules are there about your party? From this, how many different parties are possible?)
  - More models of neutral geometry than of euclidean or hyperbolic geometry (think: any model of euclidean or hyperbolic geometry must also be a model of neutral geometry)
- Taking more axioms means the new set of models is a subset of the models before taking that axiom



Johann Lambert (1728-1777)

- Introduced the idea of hyperbolic trigonometry functions
- Made many conjectures about non-Euclidean space
- Showed that the area of a hyperbolic triangle depends solely on the sum of its angle measures (the greater the area, the smaller the sum of the angles)
  - The larger the area of the triangle, the smaller the sum of its angles
  - The area of a hyperbolic triangle can be expressed only in terms of its angles, unlike Euclidean geometry, which depends on the length of its sides
- Asserted that similar triangles do not exist in hyperbolic geometry, as if the angles get larger or smaller, the area changes in such a manner that prevents the existence of them
  - Follows from the fact that only triangles with the same angles will have the same area
- Side note: also the first guy to prove π is irrational

Lobachevsky (1792-1856)

- Largely credited with the discovery of hyperbolic geometry
  - Sometimes referred to as Lobachevskian geometry
- He replaced Playfair's axiom with the statement that given a line and a point not on the line, there exists *more than one* parallel line through the point
- Developed the angle of parallelism (more on this later)
  - Depends on the distance between the line and the point
- Lobachevsky, Bolyai and Gauss all take some credit for developing non-Euclidean geometry, but they all discovered what they did on their own
  - Lobachevsky focused only on hyperbolic geometry, and did not explore other geometries
- By creating a consistent geometry without the fifth axiom, he showed that you can't prove it from the other 4

Story of Bolyai

- Bolyai was a mathematician who became obsessed with Euclid's fifth postulate
- It dominated his life so much that his father (who tried in vain to prove the 5th axiom) tried to convince him to stop
  - "For God's sake, I beseech you, give it up. Fear it no less than sensual passions because it too may take all your time and deprive you of your health, peace of mind and happiness in life"
- Bolyai did not stop though, and eventually came to the conclusion that the fifth axiom was independent of the other four
- He wrote his father to say that "I created a new, different world out of nothing."

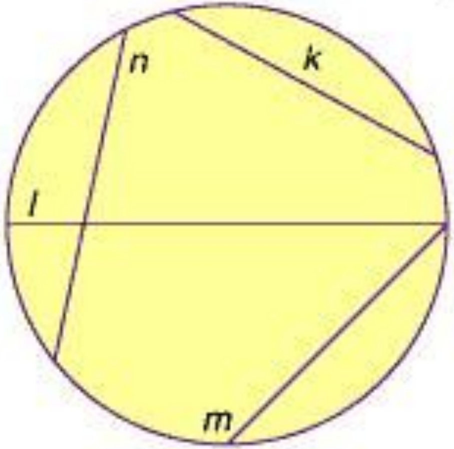
Story of Gauss

- Nobody liked Gauss
- After Bolyai completed his theories about Elliptical and Hyperbolic space, his father sent his proofs to a childhood friend, Gauss
- Gauss essentially took all the credit
  - “To praise it would amount to praising myself. For the entire content of the work... coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years.”
- Apparently this sort of thing used to happen all the time
  - Gauss in particular tended to do this a lot
    - Gauss may have come up with it first (the validity of this is unclear), but he did not publish it, probably for fear of humiliation

The Axioms of Hyperbolic Geometry

- "To draw a straight line from any point to any point."
- "To produce [extend] a finite straight line continuously in a straight line."
- "To describe a circle with any centre and distance [radius]."
- "That all right angles are equal to one another."
- “Given a line and a point not on the line, there exists more than one line through the point parallel to the given line.”
    - This is a case of the negation of Playfair’s Axiom

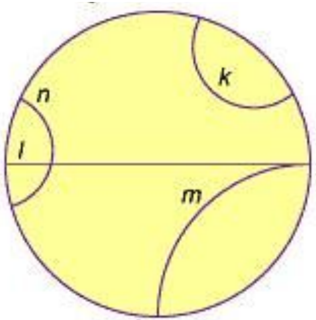
Models of Hyperbolic Geometry



Klein-Beltrami model

- Klein-Beltrami Model
    - Define “plane” to be the interior of a Euclidean circle
    - Define “point” to be any of the Euclidean points inside that circle
    - Define “line” to be a chord of that circle, excluding endpoints
        - Preserves “straightness” of lines
        - Angles become warped
    - Not really used much in practice
- Claim: under this interpretation of “point”, “line”, and “plane”, the five postulates of hyperbolic geometry can be proven true (using Euclidean geometry, of course)
    - We are creating a model of hyperbolic geometry using Euclidean structures

Models of Hyperbolic Geometry

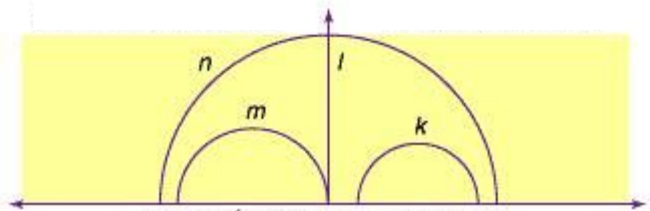


Poincaré disk model

- Poincaré Disk Model
  - Define “plane” to be the interior of a Euclidean circle
  - Define “point” to be any of the Euclidean points inside that circle
  - Define “line” to be a Euclidean arc (excluding endpoints) in the interior of the circle, orthogonal to the boundary (plus all diameters)
    - Preserves angle measures
    - Lines are no longer straight
- Claim: under this interpretation of “point”, “line”, and “plane”, the five postulates of hyperbolic geometry can be proven true (using Euclidean geometry, of course)

Models of Hyperbolic Geometry

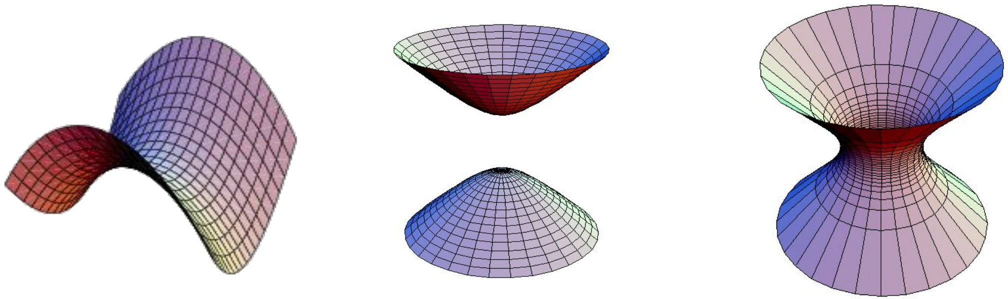
- Poincaré Half-Plane Model
  - Define “plane” to be a Euclidean half-plane (excluding the line itself)
  - Define “point” to be all the Euclidean points of that half-plane
  - Define “line” to be Euclidean semi-circles (w/o endpoints) in that half-plane such that the center lies on the line (also perpendicular rays)



Poincaré upper half-plane model

The Hyperbolic Model

- The “plane” in this model is a curved hyperboloid
  - Formed by rotating a Hyperbola
- “Lines” appear as hyperbolas
  - They are cross sections of a hyperbolic surface
- “Point” is any Euclidean point on the hyperboloid
- The hyperbolic model is one way of visualizing hyperbolic geometry
  - It is not the only way of doing so

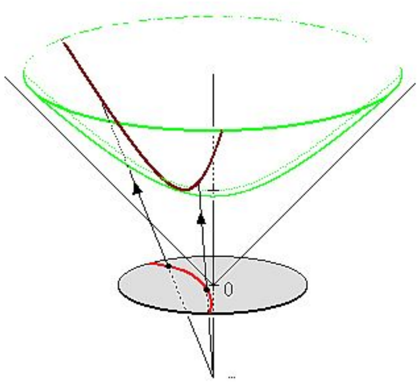


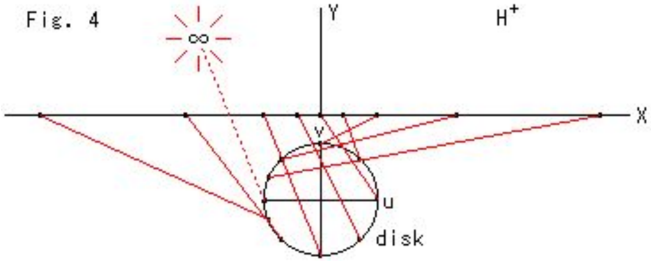
The Three Ways of Understanding "Parallel"

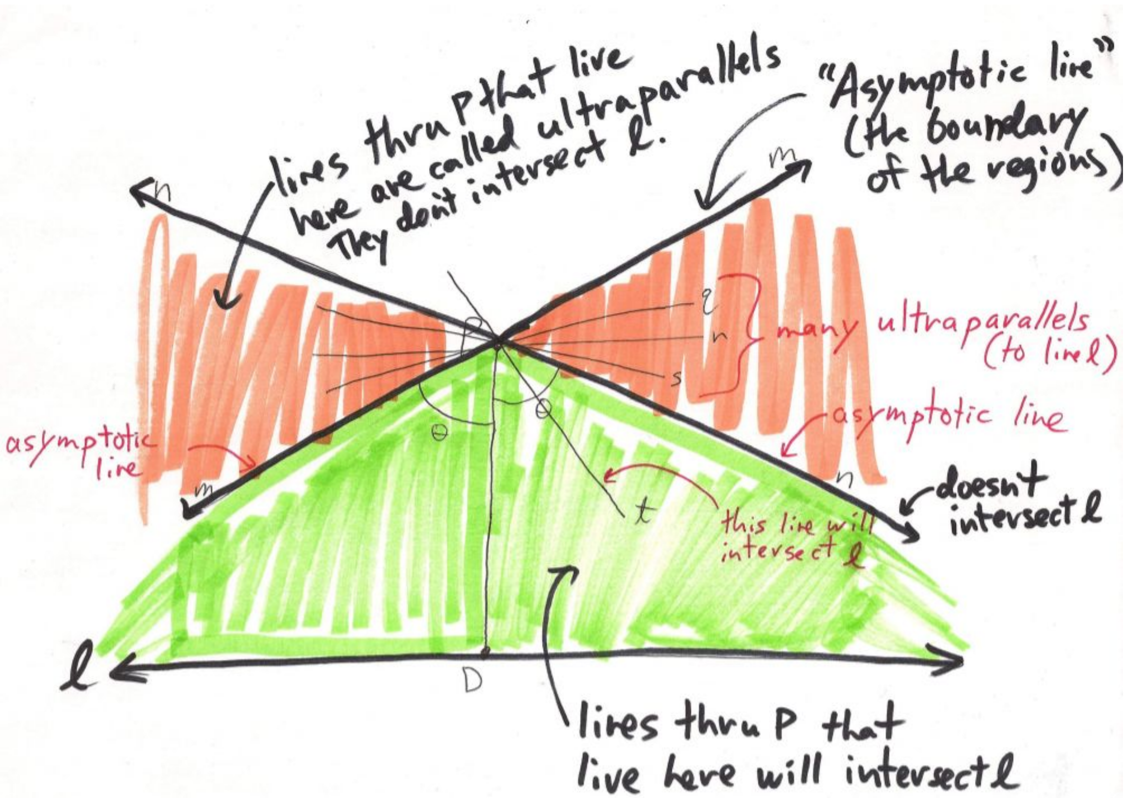
- In Euclidean geometry, if two lines are parallel, they...
  - (These are all provably equivalent, so we never have to say which we mean)
    - Have a common perpendicular
    - Never intersect
    - Are always equidistant
- In hyperbolic geometry, we need a finer distinction
  - If two lines are parallel (meaning never intersecting), they can either be "ultraparallel" or "asymptotically parallel"

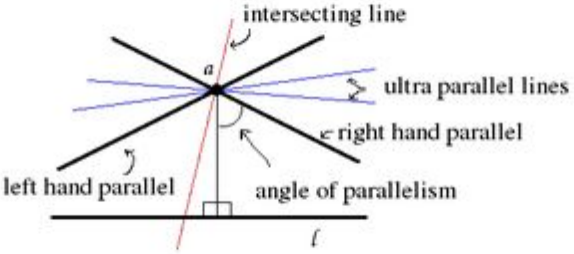
The 4 Different Ways of Modelling Hyperbolic Geometry and their Connections

- All of the models that exist in the Euclidean plane are just projections from a Euclidean Hyperboloid onto a Euclidean circle
- The Poincaré disk model unrolls into the Poincaré half-plane model



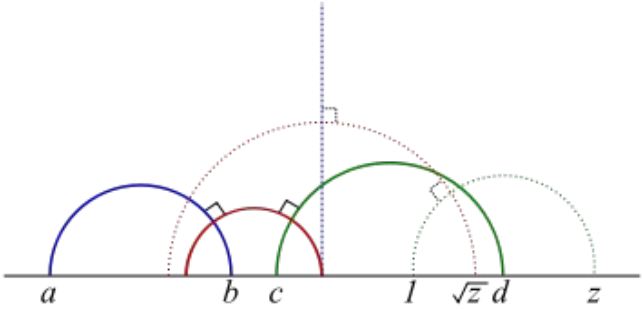


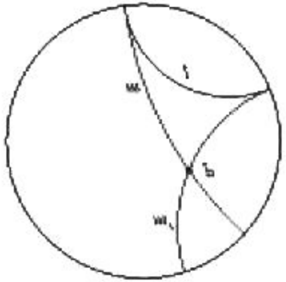

lines thru P that live here are called ultraparallels
They don't intersect l.
"Asymptotic line"
(the boundary of the regions)
many ultraparallels (to line l)
asymptotic line
asymptotic line
doesn't intersect l
this line will intersect l
lines thru P that live here will intersect l


intersecting line
ultra parallel lines
right hand parallel
left hand parallel
angle of parallelism

Ultraparallels and Asymptotic Lines

- Ultraparallel lines do not intersect and are not “limiting parallel”
  - “Limiting parallel” lines reach a point of intersection as the distance from the center reaches infinity. They appear to intersect on disk representations of hyperbolic space, but only ever approach intersection.
- Ultraparallel Theorem
  - every pair of ultraparallel lines (lines that are not intersecting and not limiting parallel) has a unique common perpendicular hyperbolic line.





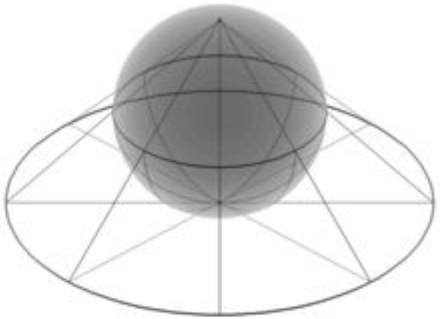
The Axioms of Elliptic Geometry

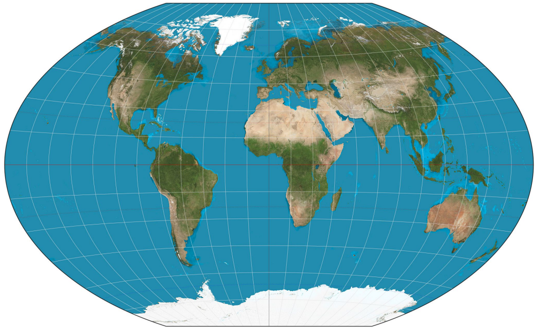
- "To draw a straight line from any point to any point."
- "To describe a circle with any centre and distance [radius]."
- "That all right angles are equal to one another."
- “Given a line and a point not on that line, there exist no lines through the point parallel to the given line.”
  - This is a case of the negation of Playfair’s axiom
  - Basically, all lines intersect

Elliptic Geometry is a subcase of Riemannian geometry

Models of Elliptic Geometry

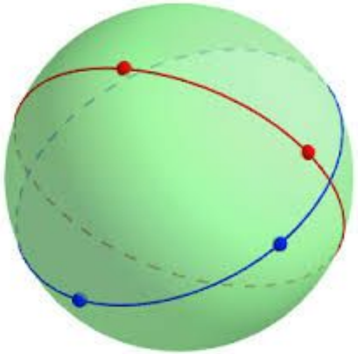
- Typically expressed through projection onto a circle such that one pole becomes the center and one pole becomes the edge
- https://en.wikipedia.org/wiki/List_of_map_projections
  - Many ways of modelling this have been achieved



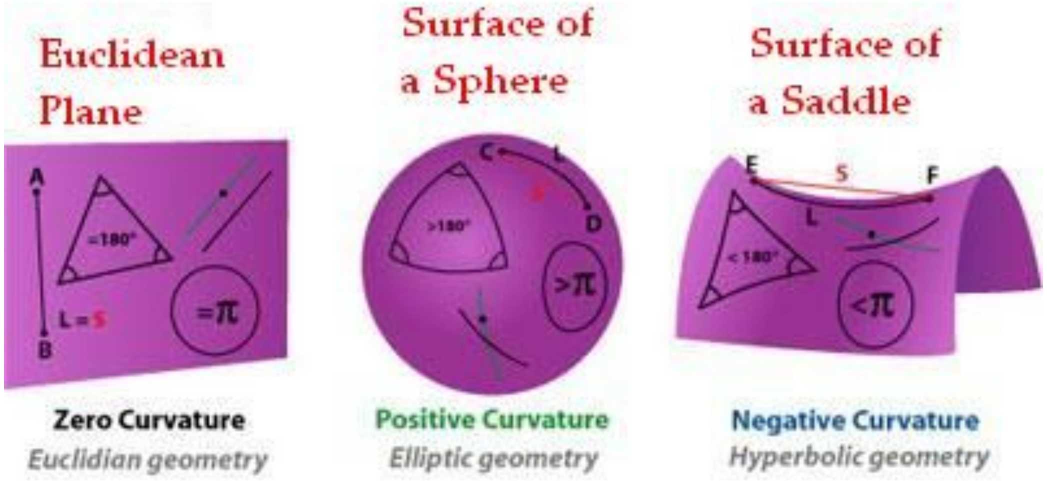


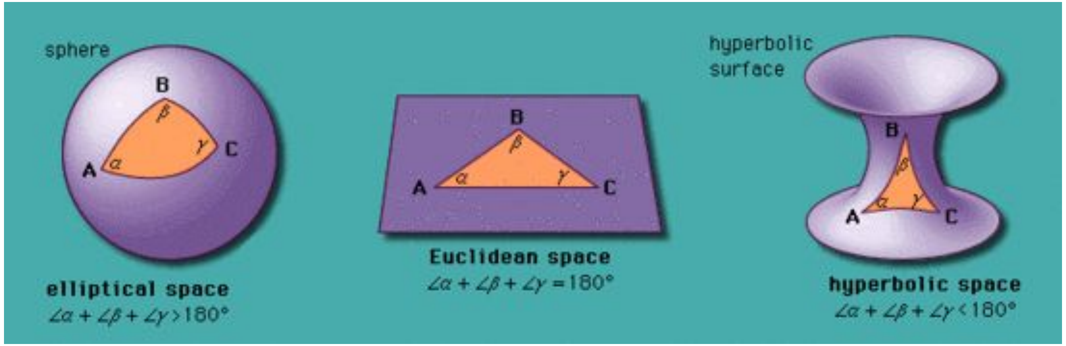
The Elliptic Model aka Spherical Model

- “Plane” now defined as a Euclidean sphere
- “Lines” defined as Euclidean “great circles” on euclidean sphere
  - “Great Circles” are circles on a sphere whose center is the center of the sphere
- “Point” is defined as any point on the Euclidean sphere
  - “Antipodal points” on the sphere are considered the same point



DIFFERENT TYPE OF GEOMETRIES





Is Our World Euclidean, Hyperbolic or Elliptic?

- Most mathematicians were entirely unsure of whether or not they believed the world is euclidean or non-euclidean
- Bolyai: “It is not possible to decide through mathematical reasoning alone if the geometry of the physical universe is Euclidean or non-Euclidean; this is a task for the physical sciences.”
  - Math really is just abstraction from the physical world, and if math suggests there exists more than one consistent geometry, humans must then decide which one is the “right” one
- Hyperbolic space and elliptical space appears Euclidean at close range

Gauss Measuring Mountains

- At one point Gauss decided to test ideas of elliptical geometry - chiefly, that the angles in a triangle could add up to more than 180°, by measuring angles on the Earth between three mountains.
- Experiment designed to show that on the surface of Earth, the angles of triangles, despite common belief, add to more than 180°
- This was essentially a last resort to try and figure out which geometry was the "right" geometry for the world we live in
    - The point of measuring between mountains as opposed to smaller triangles is that the angles of elliptic and hyperbolic triangles approach 180° as the triangle gets smaller

Consistency of Hyperbolic Geometry

- Does hyperbolic geometry, uh… work?
- Do the five axioms of hyperbolic geometry lead to a contradiction, after all, as Saccheri thought (hoped)?
- How do we see if hyperbolic geometry is consistent?
- Answer: Make a model of hyperbolic geometry in Euclidean geometry.
- Beltrami (1868): Hyperbolic geometry is consistent if and only if Euclidean geometry is

More Than One Math (?????)

- The concept that the fifth axiom was not necessary to have consistent mathematical systems really freaked people out
- We now understand that mathematical truth exists separately in any internally consistent system
  - However, since it was thought that all the axioms HAD to be true, this put enormous strain on the trust people had in mathematics, since it appeared that not even mathematics was set in stone.
  - Kant: Euclidean Geometry is a synthetic a priori truth. 19th century mathematics: No.

Mathematical Truth (Truth at all?????)

- Mathematical “truth” only exists under the assumption of certain axioms
    - Since there are ways of picking different axioms and ending up with consistent systems, this means we must then find the “right” axioms if we hope to attain this “truth” about our world
        - So, as Gauss tried to investigate, which type of world do we live in, geometrically?
- The lack of a certain mathematical “truth” also draws into question whether there is any such thing as “truth”, since math has long since been viewed as the pinnacle of human reasoning
- Does this imply that even human reasoning is simply flawed?
- Philosophical issues are clearly arising...