An Introduction to Ergodic Theory and Prime Orbits

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Dynamical Systems

Abstractly, a **dynamical system** is a set X equipped with time evolution rule.

Discrete-time systems: a map $f : X \to X$, which can be iterated.

Continuous-time systems (flows): a family of maps $\{f^t : X \to X \mid t \in \mathbb{R}\}$.

Can always be viewed as a (semi) group action of $\mathbb{Z}, \mathbb{N}, \mathbb{R}$, or $\mathbb{R}^+$ on X.
Systems usually preserve a particular structure on the set X.

Many different flavors!

Dynamics has many subdivisions:

Topological: X a topological space, f a continuous map.

Geometric: X a metric space, f an isometry.

Algebraic: X your favorite algebraic structure, f an endomorphism.

Smooth: X a smooth manifold, f a differentiable map.

Measure-theoretic: X a measure space, f a measure-preserving map.

Symbolic dynamics, arithmetic dynamics, complex dynamics, ...

Some notation, important notions

Given a map f (flow f_t), its **iterates** are $f^n = f \circ \ldots \circ f$ (n times) and we have $f^{n+m} = f^n \circ f^m$ ($f_{s+t} = f_s \circ f_t$).

Given a point $x \in X$, its **orbit** is $\mathcal{O}_+(x) = \{f^n(x) \mid n \in \mathbb{N}\}$ or $\{f_t(x) \mid t > 0\}$.

A point $x \in X$ is called **periodic** if $f^k(x) = x$ for some $k \in \mathbb{N}$ (finite orbit).

First Example

Rotations of the circle $\mathbb{T}$!

We will consider $\mathbb{T}$ to be the interval $[0, 1]$, with points 0 and 1 identified.

Consider the map $R_\alpha : \mathbb{T} \to \mathbb{T}$ by $x \mapsto x + \alpha \pmod 1$.

If $\alpha \in \mathbb{Q}$, say $\alpha = p/q$ in lowest terms, then every $x \in \mathbb{T}$ has period q.

If $\alpha \notin \mathbb{Q}$, then it turns out every point's orbit is dense!

Second Example

Linear flows on the torus $\mathbb{T}^2$!

Consider the flow $T_\alpha^t : \mathbb{T}^2 \to \mathbb{T}^2$ by $(x, y) \mapsto (x + \alpha t, y + t)$.

Notice, for $n \in \mathbb{N}$, $T_\alpha^n(x, 0) = (x + n\alpha, 0)$.

If $\alpha \in \mathbb{Q}$, say $\alpha = p/q$ in lowest terms, then every $s \in \mathbb{T}^2$ has return time q.

If $\alpha \notin \mathbb{Q}$, then it turns out every point's orbit is dense!

The context for Ergodic Theory

We will take $(X, \mathcal{B}, \mu)$ to be a Borel measure space,
$T : X \to X$ a measure-preserving transformation (or flow).

This means that for any measurable $A \subseteq X$, we have $\mu(T^{-1}A) = \mu(A)$
(or for flows, $\mu(T^{-t}A) = \mu(A)$ for each t).

Ergodic Transformations

Let $A\Delta B = (A \setminus B) \cup (B \setminus A)$ denote the symmetric difference.

A set $A \subseteq X$ is said to be **essentially T-invariant** if $\mu(T^{-1}(A)\Delta A) = 0$.

T is said to be **ergodic** if every essentially T-invariant set $A \subseteq X$ satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

It turns out that for irrational α, both R_α and T^t_α are ergodic.

Equidistribution and the Birkhoff Ergodic Theorem

Let $(X, \mathcal{B}, \mu)$ be a finite measure space.
A sequence $(a_k) \subseteq X$ is said to be **equidistributed** if for any $f \in L^1(X)$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(a_k) \overset{n\to\infty}{\longrightarrow} \frac{1}{\mu(X)}\int_X f d\mu.$$

Points whose orbits are equidistributed are called **generic**.

Theorem (Birkhoff Ergodic Theorem)

T, a measure-preserving transformation on the finite measure space $(X, \mathcal{B}, \mu)$, is ergodic with respect to μ if and only if μ-a.e. $x \in X$ is generic under T.

Unique Ergodicity

$T : X \to X$ is said to be **uniquely ergodic** if there is only one measure μ on $(X, \mathcal{B})$ with respect to which T is ergodic.

T is uniquely ergodic (with ergodic measure μ) if and only if *every* point $x \in X$ is generic with respect to T. That is, for every $x \in X$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \xrightarrow{n\to\infty} \frac{1}{\mu(X)}\int_X f d\mu.$$

Orbits along Primes

For number theoretic reasons, it is interesting to ask when, for all $x \in X$,

$$\frac{1}{\pi(n)} \sum_{p<n \text{ prime}} f(T^p x) \overset{n\to\infty}{\longrightarrow} \frac{1}{\mu(X)} \int_X f d\mu.$$

It turns out it is difficult to provide a simple criterion for this to happen. There are a few known examples, but little general theory.

My work with Dr. Adam Kanigowski provided the first examples of weak-mixing* dynamical systems for which this occurs.

Results about Prime Orbits

Vinogradov

Showed every point's orbit along primes is equidistributed for an irrational rotation of the circle.

Green-Tao (2012)

Showed every point's orbit along primes is equidistributed for nilmanifolds.

Bourgain (2013)

Showed every point's orbit along primes is equidistributed for some 3 IETs.

Reparameterizing the flow on $\mathbb{T}^2$

Let $r : \mathbb{T}^2 \to \mathbb{R}^+$ be a smooth function.
Now define $T_t^{\alpha,r} : \mathbb{T}^2 \to \mathbb{T}^2$ by the differential equations

$$\frac{dx}{dt} = \alpha \cdot r(x, y) \text{ and } \frac{dy}{dt} = r(x, y).$$

This is the same linear flow from before, but with speed rescaled by r.

We can choose α, r so that the flow is both uniquely ergodic and weak-mixing.

Methodology

For well chosen α, we can establish rigidity in flowing for times which are integer multiples of particular primes. In other words, there is a sequence of primes (q_n) for which

$$\sup_{0 \le K \le e^{\delta q_n}} \sup_{x \in \mathbb{T}} \operatorname{dist}(\, T^{\alpha,r}_{Kq_n}(x,y), (x,y)\,) \overset{n\to+\infty}{\longrightarrow} 0.$$

We then basically only have to look at the behavior of the flow in residue classes modulo the Kq_n.

Number Theory Contribution

Let $\pi(x; q, a)$ denote the number of primes less than or equal to x which are congruent to $a \pmod{q}$.

Theorem (Siegel-Walfisz Theorem)

If q is prime, then

$$\pi(x; q, a) \sim \frac{Li(x)}{q-1}.$$

This means primes are even spread among equivalence classes modulo a smaller prime q.

Critical Trick

We can turn the sum along primes into a sum along residue classes modulo the q_n. For an appropriate sequence $K_n \to +\infty$, and any $g \in L^1(\mathbb{T}^2)$, we have

$$\frac{1}{\pi(K_n q_n)} \sum_{p<K_n q_n} g(T_p^{a,r}(x,y)) = \frac{1}{\pi(K_n q_n)} \sum_{a<q_n} \left(\sum_{p=k_p q_n + a < K_n q_n} g(T_{k_p q_n + a}^{a,r}(x,y)) \right)$$

and this is roughly

$$= \frac{1}{\pi(K_n q_n)} \sum_{a<q_n} g(T_a^{a,r}(x,y)) \cdot \pi(K_n q_n; q_n, a) \sim \frac{1}{q_n - 1} \sum_{a<q_n} g(T_a^{a,r}(x,y))$$

Results

Finally, by unique ergodicity, we know that

$$\frac{1}{q_n - 1} \sum_{a < q_n} g(T_a^{\alpha,r}(x,y)) \overset{n \to +\infty}{\longrightarrow} \int_X g \, d\mu,$$

and therefore

$$\frac{1}{\pi(K_n q_n)} \sum_{p < K_n q_n} g(T_p^{\alpha,r}(x,y)) \overset{n \to +\infty}{\longrightarrow} \int_X g \, d\mu,$$

so orbits along a subsequence of primes equidistribute for every $(x,y) \in \mathbb{T}^2$. As a corollary, orbits along primes are always dense in this system.

Open Problems

Sarnak's Conjecture

Let μ denote the Mobius function. For any compact metric space X, if T is a zero entropy, continuous map on X, then for any $f \in C(X)$,

$$\frac{1}{n}\sum_{k\leq n} f(T^k x)\mu(k) \overset{n\to+\infty}{\longrightarrow} 0.$$

Chowla's Conjecture

If $a_1, \ldots, a_m$ are fixed integers (with at least one odd), then

$$\sum_{n\leq x} \mu(n+1)^{a_1} \cdot \ldots \cdot \mu(n+m)^{a_m} = o(x)$$

The End

Bonus Slide 1: Weak Mixing

A measure-preserving transformation T on $(X, \mathcal{B}, \mu)$ is **weak mixing** if, for any two $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \mu(T^{-k}(A) \cap B) - \mu(A) \cdot \mu(B) \right| = 0$$

For weak mixing transformations T, the only eigenfunctions for the corresponding unitary operator U_T on $L^2_\mu(X)$ (by $f \mapsto f \circ T$) are constant.

Bonus Slide 2: Conjugacy

Let (X, f) and (Y, g) be two dynamical systems. An invertible map $\pi : X \to Y$ is said to be a **conjugacy** between the systems if $\pi \circ f^t = g^t \circ \pi$ for all t.

This gives an equivalence relation on dynamical systems, and preserves all important aspects of the dynamics.

(Conjugacies are to dynamics what isomorphisms are to algebra, homeomorphisms to topology, etc.)

Bonus Slide 3: Suspension flows

Given a dynamical system (X, T) and a function $f : X \to \mathbb{R}^+$, consider the space

$$X_f = \{(x, t) \in X \times \mathbb{R}^+ \mid 0 \leq t \leq f(x)\} / \sim$$

where $\sim$ is the equivalence relation given by $(x, f(x)) \sim (Tx, 0)$.

We can define a flow on this space by flowing vertically from any point with unit speed, then upon reaching the ceiling of the function, jumping back down to the point with which it is identified, flowing vertically with unit speed, etc.