Embedded Cobordisms

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Background in the Abstract Case

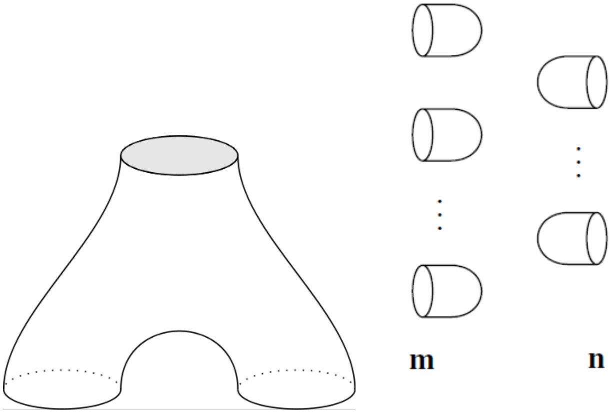
- Manifolds are assumed to be smooth and oriented

Definition

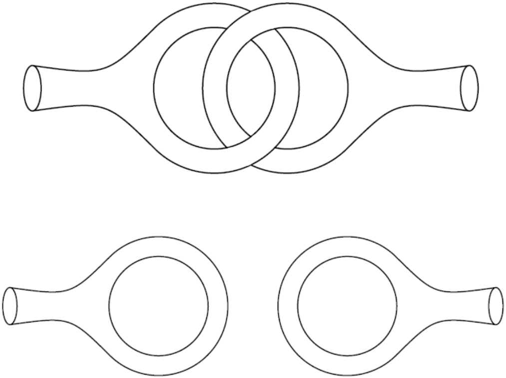
Let M, N be closed $(n-1)$-manifolds. Then a cobordism from M to N is a compact oriented n-manifold whose boundary is $M \sqcup N$.

- This definition treats smooth manifolds *abstractly*
- Our cobordisms have an *in-boundary* and *out-boundary*
- Equivalence is basically up to diffeomorphism
- Empty manifold counts!

Examples of Cobordisms

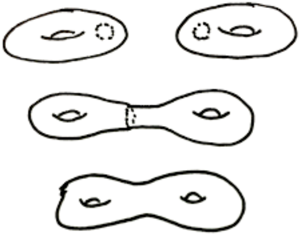


Another Example



Background

- Can compose cobordisms:



Categories

Categories are a structure with

- a collection of objects and arrows
- an associative composition operation
- identity arrows for every object

Examples:

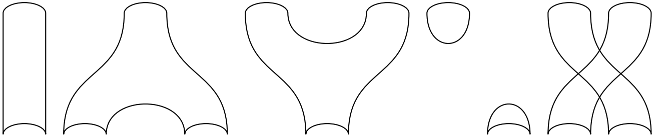
- **Set**
- **Grp**
- $\mathbf{Vect}_{\mathbf{K}}$

$$\begin{tikzcd}
A \arrow[loop left, "1_A"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[loop right, "1_B"] \arrow[d, "g"] \\
 & C \arrow[loop right, "1_C"]
\end{tikzcd}$$

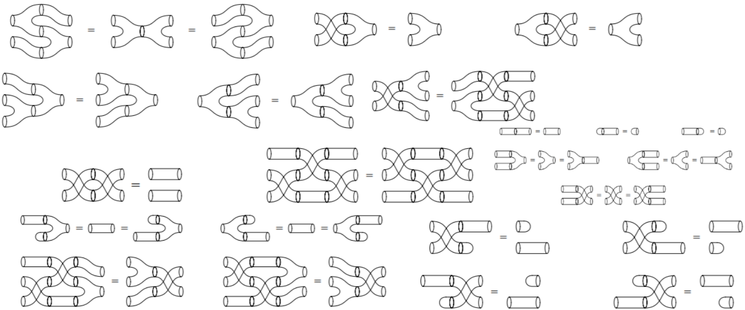
We can turn 2D cobordisms to build a category **2Cob**

- Objects: closed 1-manifolds (disjoint unions of circles)
- Morphisms: (equivalence classes of) 2D cobordisms

- Generators:



Relations

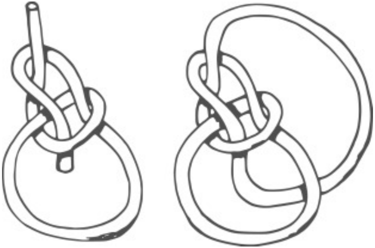


The Embedded Case

- For every n, let C_n be a set of fixed circles in the plane such that their projections to the x-axis are pairwise disjoint.
- An embedded cobordism C is a smooth compact oriented manifold with boundary embedded in $\mathbb{R}^2 \times [0, 1]$ such that $C \cap \mathbb{R}^2 \times \{0\} = C_n = \partial_{\text{in}}(C)$ and $C \cap \mathbb{R}^2 \times \{1\} = C_m = \partial_{\text{out}}(C)$ for some integers m, n
- Two cobordisms are equivalent if there exists an ambient isotopy fixing the boundary taking the first cobordism to the second.

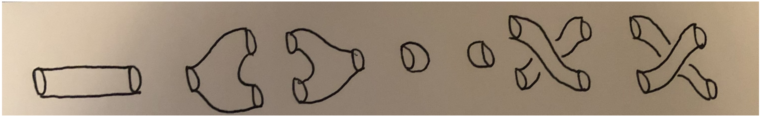
The Embedded Case Cont.

- Can have ‘knots’.



Generators

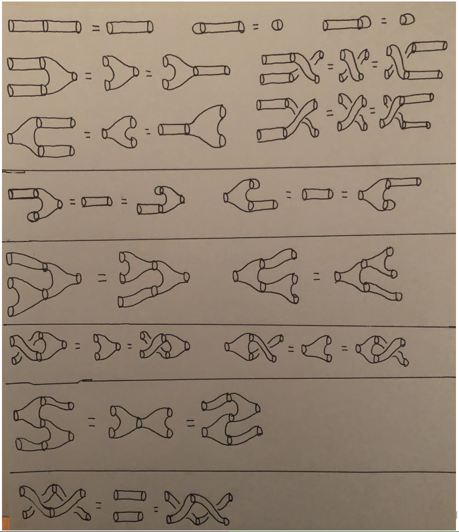
Conjecture: Embedded Cobordisms are generated by the following:



Relations

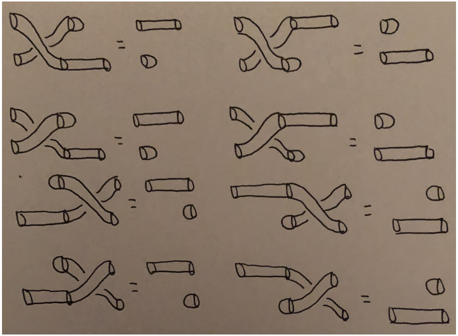
Most basic relations

Relations most similar to abstract case

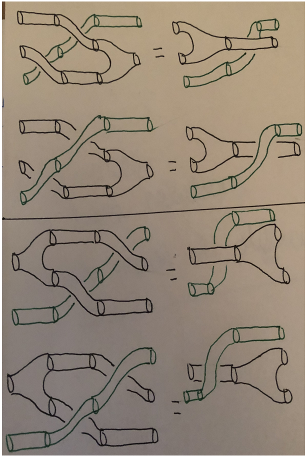


Relations-How crossings make a difference

Having different twists causes more relations to occur

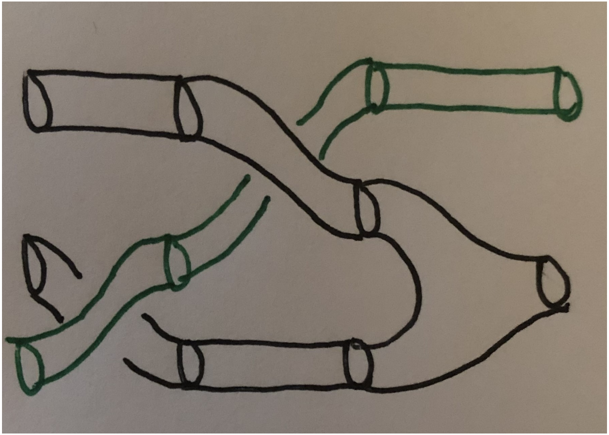


Dragging a strand past the waist of a pair of pants



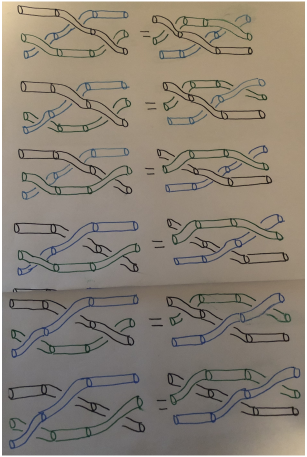
Crossings with pairs of pants

Knotted relation



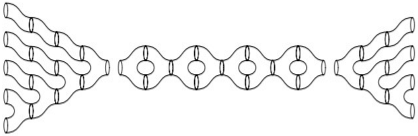
Braid Group Relation

Analogous to Reidemeister 3



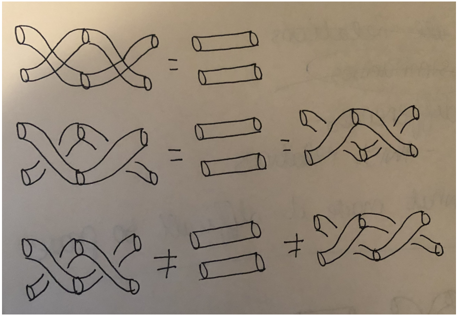
Difficulty with proving sufficiency

Abstract case uses a normal form



Difficulty with proving sufficiency

Abstract proof relies on elimination of twists



Proof ideas and difficulties

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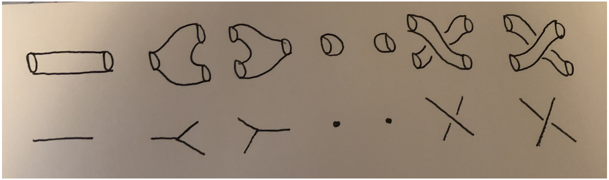
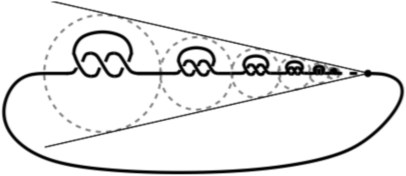
- tracking critical points with Morse functions
- moving cobordisms into general position
- create some sort of normal form
  - isolate twists
  - use what we know from knot theory

Generators in 1D

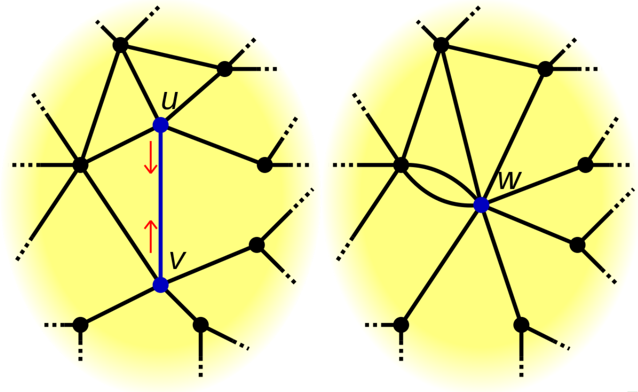


- Strategy: Associate a cobordism with a 1D diagram, define isotopies of diagram, and show ambient isotopies of the 2-d case correspond to isotopies of the diagram.
- Piecewise linear to avoid pathologies like this:

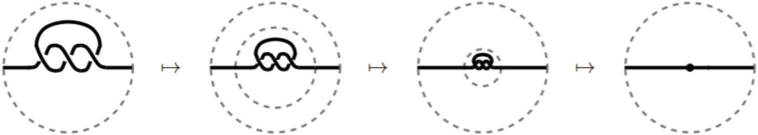


Isotopies of 1D

- Need to find definition of isotopy that corresponds to 2-d case.
- Need to be able to contract edges and move vertices past each other.
- Solution in knot theory is ambient isotopy but that doesn't work because of the above.



- Also need to avoid this.



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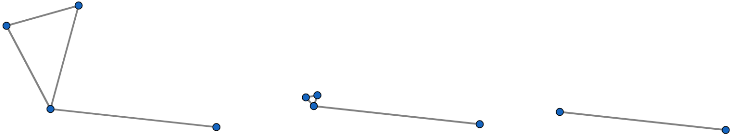

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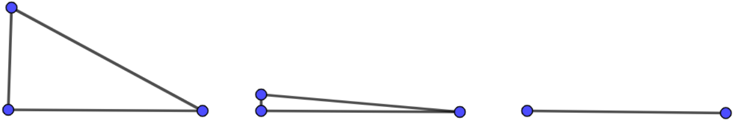
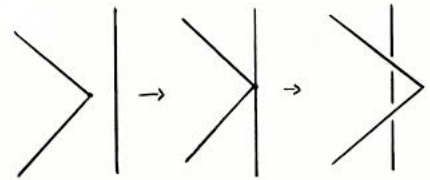


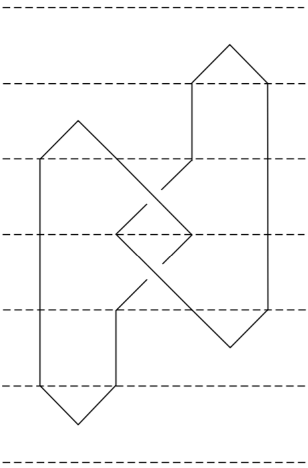
- Solution: Define isotopy by metrizing embedded graphs and demand that there are no ajacent edges that get arbitrarily small.
- The above definition turns the problem to a manageable analysis problem.
- Metric:

$$d_{1d2dC}(G,H) =$$
$$\inf_{f:G\to H}\left(\sup_{x,y\in G}|d(x,y)-d(f(x),f(y))| + \sup_{x\in G}||x-f(x)||_{L^2}\right)$$
$$+$$
$$\inf_{f:H\to G}\left(\sup_{x,y\in H}|d(x,y)-d(f(x),f(y))| + \sup_{x\in H}||x-f(x)||_{L^2}\right)$$



Generators in 1D

- Easy to prove generators
- Just isotopy diagram to general position and decompose based on vertices



Relations in 1D

- Strategy: as in the above, give a deterministic way to break diagram up into generators.
- Given an isotopy, show that the diagram changes locally by the relations.
- This is done by going through cases of what diagram looks like locally and narrow down what can happen locally in the isotopy by using a lot of delta-epsilon arguments.

Generators and relations seems very reminiscent of algebra...

Conversion between our topological setting and an algebraic one?

This is accomplished via a TQFT: a symmetric monoidal functor from the category of cobordisms to the category of vector spaces.

TQFTs have the structure of a Frobenius algebra, due to the conditions we will see must be satisfied.

The Correspondence

The objects in our categories:

Collections of Circles $\rightarrow$ Vector Spaces over a Field

The morphisms:

Embedded Cobordisms $\rightarrow$ Linear Maps

Compatibility

The functor should respect composition and disjoint union of cobordisms. In particular if $S^1 \mapsto V$, then

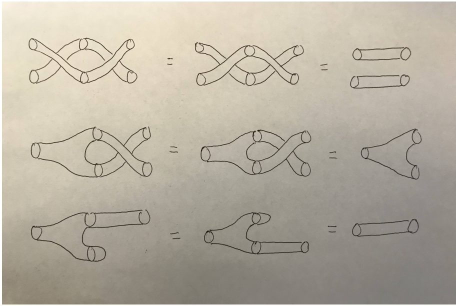
$$S^1 \sqcup S^1 \mapsto V \otimes V \text{ and so } \bigsqcup_{i=1}^{n} S^1 \mapsto \bigotimes_{i=1}^{n} V$$

And if C_1, C_2 are composable cobordisms, then

$$C_1 \mapsto A, C_2 \mapsto B \text{ implies } C_1 \circ C_2 \mapsto A \circ B$$

Satisfying the Relations

Difficulty: the TQFT has to respect our relations in order to be constant on equivalence classes.



Defining a TQFT for the abstract case turns out to be manageable

However, in the abstract setting, there is only one crossing. In particular, it corresponds to the map sending $a \otimes b \mapsto b \otimes a$.

We could always define a TQFT for our setting in the same way, but at the cost of losing the desired complexity

We want to have these crossings give distinct linear maps

Equations to be Satisfied

1. $O \circ U = U \circ O = I$
2. $(I \otimes C') \circ P = (C' \otimes I) \circ P = I$
3. $P' \circ (I \otimes C) = P' \circ (C \otimes I) = I$
4. $P' \circ (I \otimes P') = P' \circ (P' \otimes I)$
5. $P' \circ O = P' = P' \circ U$
6. $O \circ P = P = U \circ P$
7. $(P' \otimes I) \circ (I \otimes P) = P \circ P' = (I \otimes P') \circ (P \otimes I)$
8. $(U \otimes I) \circ (I \otimes U) \circ (U \otimes I) = (I \otimes U) \circ (U \otimes I) \circ (I \otimes U)$
9. $(O \otimes I) \circ (I \otimes O) \circ (O \otimes I) = (I \otimes O) \circ (O \otimes I) \circ (I \otimes O)$
10. $(P \otimes I) \circ U = (I \otimes U) \circ (U \otimes I) \circ (I \otimes P)$
11. $(P \otimes I) \circ O = (I \otimes O) \circ (O \otimes I) \circ (I \otimes P)$

Impossible if V is One Dimensional

Sketch of proof:

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- Pants composed with cap union a cylinder is the identity, so pants cannot be the zero map
- Pants composed with either twist is the same as pants
- Hence the twists must be multiplication by 1, i.e. the identity

What else can be done?

- Trying higher dimensional spaces? Still remains difficult
- Proving it is impossible to get something nontrivial? Vector spaces might be inherently too symmetric
- Mapping to different algebraic structures altogether?

Acknowledgements

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