A Summary of Exponential Mixing for Skew Products

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The Paper

This presentation will summarize the results and methods in *Exponential Mixing for Skew Products with Discontinuities*, by Drs. Oliver Butterley and Peyman Eslami. We will also discuss this work's relationship to the broader questions at stake and other recent developments in establishing rates of mixing for certain dynamical systems.

The Main Result

Theorem

Let $F : \mathbb{T}^2 \to \mathbb{T}^2$ be given by $(x, u) \mapsto (f(x), u + \tau(x))$. Assume F is piecewise-$\mathcal{C}^2$, with $f : \mathbb{T}^1 \to \mathbb{T}^1$ uniformly expanding and covering. Either F mixes exponentially or τ is cohomologous (via a Lipschitz function) to a piecewise constant, discontinuous only where one of f, τ is discontinuous.

Similar Results for other Systems

Liverani (2004)

Proved there is exponential decay of correlations for $\mathcal{C}^4$ contact Anosov flows.

Avila, Gouëzel, and Tsujii (2005)

Showed correlations decay exponentially for certain skew products (of affine contractions) on $S^1 \times \mathbb{R}$.

Dolgopyat (2002)

Establishes rapid mixing for skew actions on compact group extensions and provides additional conditions for exponential mixing.

Formalization

By the conditions on f, we are guaranteed there is a F-invariant probability measure on $\mathbb{T}^2$, call it μ. Given $g, h : \mathbb{T}^2 \to \mathbb{C}$, define the correlation between them as

$$\mathrm{Cor}_{g,h}(n) = \int_{\mathbb{T}^2} g \cdot h \circ F^n d\mu - \int_{\mathbb{T}^2} g d\mu \cdot \int_{\mathbb{T}^2} h d\mu$$

The main theorem says that either:

(i) there is $\zeta > 0$ such that $\forall g, h : \mathbb{T}^2 \to \mathbb{C}$ which are Hölder continuous, there is $C_{g,h} > 0$ for which $|\mathrm{Cor}_{g,h}(n)| \leq C_{g,h} e^{-n\zeta}$ for every $n \in \mathbb{N}$

(ii) There is a Lipschitz $\theta : \mathbb{T}^1 \to \mathbb{R}$ and a piecewise constant $\chi : \mathbb{T}^1 \to \mathbb{R}$ such that $\tau = \theta \circ f - \theta + \chi$, and χ is only discontinuous where F is.

Transversality

We are free to assume that $\inf f' > 2$ (by iterating if necessary), and it follows that the constant conefield with cones $\mathcal{K} = \{ \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right) \mid |\frac{\beta}{\alpha}| \leq \frac{2 \sup |\tau'|}{\inf f' - 1} \}$ is strictly invariant under the map $DF(x) = \begin{pmatrix} f'(x) & 0 \\ \tau'(x) & 1 \end{pmatrix}$.

Let $x_1, x_2 \in \mathbb{T}^1$ be given such that $f^n(x_1) = f^n(x_2) = y$. These points are said to be transversal (denoted $x_1 \pitchfork x_2$) if $DF^n_{x_1} \mathcal{K} \cap DF^n_{x_2} \mathcal{K} = \{0\}$.

The Quantity $\varphi(n)$

Let the Jacobians be denoted by $J_n = |(f^n)'|^{-1}$. We then define

$$\varphi(n) = \sup_{y \in \mathbb{T}^1} \sup_{x_1 \in f^{-n}(y)} \sum_{\substack{x_2 \in f^{-n}(y) \\ x_1 \not\pitchfork x_2}} J_n(x_2)$$

This quantity provides control over the proportion of preimage points which are not transversal. We will see that the dichotomy for the main result arises by analyzing the behavior of $\varphi(n)$ as n grows.

If $\varphi(n)$ does Not Decay

Suppose that $\limsup_{n\to\infty} \varphi(n)^{1/n} = 1$.

Under this condition, it can be shown that for every $n \in \mathbb{N}, y \in \mathbb{T}^1$ there is a $L_n(y) \in \mathbb{RP}^1$ such that for each $x \in f^{-n}(y)$, we have that $L_n(y) \subset DF^n(x)\mathcal{K}$. It follows that there is $l : \mathbb{T}^1 \to \mathbb{R}$ (of bounded variation) such that $DF(x)\binom{1}{l(x)} = f'(x)\binom{1}{l\circ f(x)}$ and so $\tau' = f' \cdot l \circ f - l$.

Now, define $\theta(y) = \int_0^y l(x)dx$, and this θ is differentiable with derivative of bounded variation (hence Lipschitz) and satisfies $\tau - \theta \circ f + \theta = \chi$ for some piecewise constant χ (the derivative of the left must be zero as long as τ, f are differentiable).

Turning the Focus

We have now seen that if $\varphi(n)$ does not decay, the function τ will be cohomological to a piecewise constant. From this point on, we will assume to the contrary that $\limsup_{n\to\infty} \varphi(n)^{1/n} < 1$, and we will need to show the correlations decay exponentially.

This is where the real meat is, and perhaps where inspiration can be found for working with other systems.

The Perron-Frobenius Operator

Let $g, h : \mathbb{T}^2 \to \mathbb{C}$ be given, and assume without loss of generality that g is mean zero. Let $\hat{g}_b, \hat{h}_b$ denote their respective Fourier coefficients in the fibre coordinate, meaning that $g(x,u) = \sum_{b \in \mathbb{Z}} \hat{g}_b(x) e^{-ibu}$ (likewise for h). Define the twisted transfer operator by

$$\mathcal{L}_b^n g(y) = \sum_{x \in f^{-n}(y)} J_n(x) \cdot g(x) \cdot e^{ib\tau_n(x)}$$

simple manipulation (and a change of variables) then shows

$$\int_{\mathbb{T}^2} (g \cdot h \circ F^n)(x,u)\, dx du = \sum_{b \in \mathbb{Z}} \int_{\mathbb{T}^1} \mathcal{L}_b^n \hat{g}_b(x) \cdot \hat{h}_b(x) dx$$

What to Estimate and How

Gaining control over $|\int_{\mathbb{T}^1} \mathcal{L}_b^n \hat{g}_b(x) \cdot \hat{h}_b(x) dx|$ will be crucial in order to show exponential mixing. This will be done by controlling the size of $\mathcal{L}_b^n \hat{g}_b$.

We will work with functions of bounded variation (for suitability of dealing with discontinuities), which form a banach space with respect to the norm $\| \cdot \|_{BV} = \text{Var}(\cdot) + \| \cdot \|_{L^1}$. For future convenience, also define the equivalent norm $\| \cdot \|_{(b)} = \frac{1}{1+|b|} \| \cdot \|_{BV} + \| \cdot \|_{L^1}$.

Estimates will be obtained with respect to $\| \cdot \|_{BV}, \| \cdot \|_{(b)}$ and then the existence of a compact embedding $BV \hookrightarrow L^1(\mathbb{T}^1)$ will be leveraged to imply $\mathcal{L}_b : BV \to BV$ has spectral radius ≤ 1 (and essential spectral radius strictly less than 1).

The Lasota-Yorke Inequality

Lemma (Lasota-Yorke Inequality)

There is $\lambda > 0, C_\lambda > 0$ *such that for any* $n \in \mathbb{N}, b \in \mathbb{R}, g \in BV$*,*

$$\|\mathcal{L}_b^n g\|_{BV} \leq C_\lambda \lambda^{-n} \|g\|_{BV} + C_\lambda (1 + |b|) \|g\|_{L^1}$$

or, equivalently,

$$\|\mathcal{L}_b^n g\|_{(b)} \leq C_\lambda \lambda^{-n} \|g\|_{(b)} + C_\lambda \|g\|_{L^1}$$

This statement and its proof are relatively standard but for the factor of $|b|$ in front of the L^1 norm.

Immediate Corollary

Corollary

Pick n_0 *large enough so that* $C_\lambda \lambda^{-n_0} + \frac{1}{2} \leq \frac{3}{4}$ *and then if* $g \in BV$ *satisfies* $2C_\lambda(1+|b|)\|g\|_{L^1} \leq \|g\|_{BV}$*, we have*

$$\|\mathcal{L}_b^{n_0} g\|_{(b)} \leq \frac{3}{4}\|g\|_{(b)}$$

The issue is now that the condition on this corollary is not necessarily invariant under the operator, so we are not free to iterate this estimate in order to obtain the desired result.

The Other Case

We can now restrict our attention to situations where $2C_\lambda(1+|b|)\|g\|_{L^1} > \|g\|_{BV}$. In this case, a simple calculation using the Lasota-Yorke inequality demonstrates that we only need to obtain exponential contraction of the L^1 norm of $\mathcal{L}_b^n g$:

$$\|\mathcal{L}_b^{2n} g\|_{(b)} \leq C_\lambda \lambda^{-n} \|\mathcal{L}_b^n g\|_{(b)} + C_\lambda \|\mathcal{L}_b^n g\|_{L^1} \leq 2C_\lambda^2 \lambda^{-n} \|g\|_{(b)} + C_\lambda \|\mathcal{L}_b^n g\|_{L^1}$$

Unfortunately, this turns out to be considerably more technical and difficult to obtain than the previous estimates.

The Key Estimate

This is the key estimate whose proof makes use of the assumption about the long-term behavior of the quantity $\varphi(n)$, and there is a long discourse in the paper concerning technicalities herein.

Lemma

Let the collection of intervals $\{H_l\}_l$ *be a partition of* $\mathbb{T}^1$ *into subintervals of equal length (this length is specified, dependent on several constants). Let* $n(b), b_0$ *be sufficiently large. Then there exists* $C' > 0, \gamma' > 0$ *such that for every* $|b| \geq b_0, l$*, we have that*

$$\|\mathcal{L}_b^{n(b)} \mathbf{1}_{H_l}\|_{L^1} \leq C' e^{-n(b)\gamma'} \|\mathbf{1}_{H_l}\|_{L^1}$$

This lemma will suffice because we can take a function g_b constant on each H_l which approximates g exponentially well (as n grows) in the L^1 norm.

Proof of the Estimate

For any H_l, there is a collection of appropriately defined functions h_j (essentially $(f^{n(b)})^{-1}$ on a small interval), connected intervals ω_j for which

$$\|\mathcal{L}_b^n \mathbf{1}_{H_l}\|_{L^1} = \int_{\mathbb{T}^1} \left| \sum_j (J_n \cdot e^{ib\tau_n}) \circ h_j(z) \cdot \mathbf{1}_{f^n \omega_j}(z) \right| dz$$

and now this integral can be bounded by the square root of a sum of integrals of the form $\int_{I_p} (K_{j,k} \cdot e^{ib\theta_{j,k}})(z)dz$ (where the I_p's are small intervals partitioning $\mathbb{T}^1$, $K_{j,k} = J_n \circ h_j \cdot J_n \circ h_k$, and $\theta_{j,k} = \tau_n \circ h_j - \tau_n \circ h_k$). The key point is that now we can leverage transversality to show that this $\theta_{j,k}$ has large derivative, which (after a change of variables and integrating by parts) yields

$$\left| \int_{I_p} (K_{j,k} \cdot e^{ib\theta_{j,k}})(z)dz \right| \leq \frac{1}{|b|} C_{j,k,p}$$

Another Immediate Corollary

By applying the previous estimate and using an approximation argument, we obtain that

$$\|\mathcal{L}_b^{n(b)} g\|_{L^1} \leq C'' e^{-n(b)\gamma''} \|g\|_{L^1}$$

for some suitable constants C'', γ''. And now using this in unison with the preceding estimate of $\|\mathcal{L}_b^{2n} g\|_{(b)}$, we obtain that

$$\|\mathcal{L}_b^{2n(b)} g\|_{(b)} \leq C''' e^{-n(b)\gamma'''} \|g\|_{(b)}$$

again for some suitable constants C''', γ'''.

Bounding the Operator

We will now apply each of the corollaries from the technical estimates:

$$\text{If } 2C_\lambda(1+|b|)\|g\|_{L^1} \leq \|g\|_{BV}, \text{ then } \|\mathcal{L}_b^{n_0} g\|_{(b)} \leq \frac{3}{4}\|g\|_{(b)}$$

and

$$\text{If } 2C_\lambda(1+|b|)\|g\|_{L^1} > \|g\|_{BV}, \text{ then } \|\mathcal{L}_b^{2n(b)} g\|_{(b)} \leq C''' e^{-n(b)\gamma'''}\|g\|_{(b)}$$

By applying either estimate when appropriate (depending on which condition is satisfied), we conclude that

Lemma

There exists $b_0 > 0, \gamma > 0$ such that if $|b| \geq b_0$ and $n(b)$ is sufficiently large, we have

$$\|\mathcal{L}_b^{n(b)}\|_{(b)} \leq e^{-n(b)\gamma}$$

The Last Calculation

We first recall that

$$\int_{\mathbb{T}^2} (g \cdot h \circ F^n)(x,u)\, dxdu = \sum_{b \in \mathbb{Z}} \int_{\mathbb{T}^1} \mathcal{L}_b^n \hat{g}_b(x) \cdot \hat{h}_b(x) dx$$

The above lemma implies that there is some $\alpha \in (0,1)$ for which $\|\mathcal{L}_b^n\|_{(b)} \leq |b|^\alpha e^{-n\gamma}$ for every $n \in \mathbb{N}, |b| \geq b_0$. But now we observe

$$\left| \int_{\mathbb{T}^1} \mathcal{L}_b^n \hat{g}_b(x) \cdot \hat{h}_b(x) dx \right| \leq 2 \|\mathcal{L}_b^n\|_{(b)} \|\hat{g}_b\|_{BV} \|\hat{h}_b\|_{BV}$$

since the BV norm dominates the sup norm. And we now have that

$$\sum_{|b| \geq b_0} \|\mathcal{L}_b^n\|_{(b)} \|\hat{g}_b\|_{BV} \|\hat{h}_b\|_{BV} \leq \sum_{|b| \geq b_0} |b|^{-(2-\alpha)} e^{-n\gamma} \|\hat{g}_b\|_{\mathcal{C}^1} \|\hat{h}_b\|_{\mathcal{C}^1}$$

For terms $|b| < b_0$, we can just use quasi-compactness since f is mixing.

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