M 392C - Geometry of Locally Compact Groups: Homogeneous Spaces

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April 2024

1 The modular function

- What it is

Given λ a left-Haar measure on G, define the measure $\lambda_x(E) = \lambda(Ex)$, and λ_x is again a left Haar measure, so there is a number $\triangle(x)$ for which $\lambda = \triangle(x) \cdot \lambda_x$. The function $G \to \mathbb{R}^+$ by $x \mapsto \triangle(x)$ is called the modular function for G.

- Basic properties

The modular function is actually a continuous homomorphism from $G \to \mathbb{R}^+_\times$.

Prop If $K \leq G$ is compact, then $\triangle|_K \equiv 1$.
Proof: $\triangle(K)$ is a compact subgroup of $\mathbb{R}^+_\times$.

- Unimodular groups

G is said to be unimodular if any left Haar measure is also a right Haar measure. This is easily seen to be equivalent to saying $\triangle \equiv 1$.

A bunch of groups are unimodular: Abelian groups, discrete groups, compact groups

Non example: affine transformations

2 Homogeneous spaces

- Left group actions - G-spaces (transitive) - Example of G/H - This is pretty much all the examples

Proposition If S is a transitive G-space, pick $s_0 \in S$, and define $\varphi : G \to S$ by $\varphi(x) = xs_0$. Define $H = \text{stab}(s_0) = \{x \in G \mid xs_0 = s_0\}$. Then $H \leq G$ is a closed subgroup and φ a continuous surjection of G onto S, constant on left cosets of H. In other words, φ descends to a continuous bijection $\Phi : G/H \to S$ (with $\Phi \circ q = \varphi$, where $q : G \to G/H$ is the natural quotient map).
Remark: this is not always a homeomorphism, as the inverse map may not be continuous (consider $G = \mathbb{R}$ under the discrete topology acting on R with the usual topology by translations.

Recall a topological space is σ-compact if it is the union of countably many compact subsets.

Proposition If G is σ-compact and S is a transitive G-space, then there is a closed subgroup $H \leq G$ for which $S \cong G/H$.

Definition A *homogeneous space* is a transitive G-space S which is isomorphic to a quotient G/H (i.e. homeomorphic).

- Note the dependence of the identification on choice of basepoint - corresponds to changing the subgroup by an inner automorphism.

Big question: is there a G-invariant radon measure on G/H?

Answer Yes, as long as $\triangle_G|_H = \triangle_H$. This necessarily happens whenever G is unimodular, or whenever H is compact.

Note: the above doesn't always happen. Consider affine transformations acting on $\mathbb{R}$ by $x \mapsto ax + b$ (this is a homogeneous space), but the only measure up to scalar multiplication invariant under $x \mapsto x + b$ is Lebesgue measure, but this isn't invariant under scalings $x \mapsto ax$.

The P-function. Let G be locally compact with left Haar measure dx, H a closed subgroup with left Haar measure $d\xi$. There is a map $P : C_c(G) \to C_c(G/H)$ by $Pf(xH) = \int_H f(x\xi)d\xi$. This is well-defined by left invariance of the Haar measure $d\xi$ on H. The map P is a continuous surjection.

3 Quasi-invariant measures

Suppose μ a radon measure on G/H, then for $x \in G$ define μ_x by $\mu_x(E) = \mu(xE)$. μ is *quasi-invariant* if the measures μ_x are all equivalent (i.e. mutually absolutely continuous), and *strongly quasi-invariant* if there is a map $\lambda : G \times G/H \to (0, \infty)$ for which $d\mu_x(p) = \lambda(x, p)d\mu(p)$. In other words, strong quasi-invariance means not only are the μ_x equivalent, but that the Radon-Nikodym derivatives $\frac{d\mu_x}{d\mu}(p)$ is jointly continuous in x and p.

4 Symmetric spaces

In the case of G being a Lie group, G/H naturally has the structure of a smooth manifold and G acts by diffeomorphisms. On these spaces, there are always strong quasi-invariant measures.

Symmetric spaces have a very rich structure and their geometry is quite well-studied. Symmetric spaces have also been classified due to the classification of semisimple Lie groups.

5 ρ-functions

A ρ-function for the pair (G, H) is a continuous function $\rho : G \to \mathbb{R}^+$ for which

$$\rho(x\xi) = \frac{\triangle_H(\xi)}{\triangle_G(\xi)}\rho(x).$$

Prop If G is locally compact and H a closed subgroup, then (G, H) admits a ρ function.

Proof: define

$$\rho(x) = \int_H \frac{\triangle_H(\eta)}{\triangle_G(\eta)} f(x\eta) d\eta$$

where f is continuous and satisfies 1) $\{y \mid f(y) > 0\} \cap xH \neq \emptyset$ for all $x \in G$; 2) $\mathrm{supp}(f) \cap KH$ is compact for every compact $K \subseteq G$.

Here is the main important result from considering ρ-functions.

Theorem Given any ρ-function for the pair (G, H), there is a strongly quasi-invariant measure μ on G/H for which

$$\int_{G/H} Pf d\mu = \int_G f(x)\rho(x)dx,$$

for any $f \in C_c(G)$. μ also satisfies

$$\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{y}$$

for any $x, y \in G$.

6 Some last results

Prop If μ is a quasi-invariant measure on G/H, then $\mu(U) > 0$ for every nonempty open U.

Theorem Every strongly quasi-invariant measure on G/H arises from a ρ-function, and all such measures are strongly equivalent.