M 390C - Arithmetic Groups: Ratner's Theorem on Unipotent Flows

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1 Motivation, Statement of the Theorem

Throughout the talk, G is a linear, semisimple Lie group with finitely many connected components, and $\Gamma \leqslant G$ is a lattice. We start with an example to motivate what we will generalize.

Example

Consider linear flow lines on the torus. That is, let V be a 1-dimensional vector subspace of $\mathbb{R}^2$, and let $x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and consider the coset $V + x$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denote the natural projection map. There is a dichotomy:

1. If the slope of V is rational, then $\pi(V + x)$ is a 1-dimensional submanifold on $\mathbb{T}^2$
2. If the slope is irrational, then $\pi(V + x)$ is dense in $\mathbb{T}^2$, hence $\overline{\pi(V + x)} = \mathbb{T}^2$.

Furthermore, this picture generalizes to higher dimensions. If instead $V \subseteq \mathbb{R}^n$ is a k-dimensional vector subspace, then $\overline{\pi(V + x)}$ is homeomorphic to a torus (of dimension $\leq n$).

We now develop a wide-ranging generalization of the above phenomenon. First, recall:

Definition 1.0.1: Unipotent Elements

If $G \leqslant \mathbf{SL}(\ell, \mathbb{R})$, then $g \in G$ is said to be *unipotent* if its only eigenvalue is 1. Equivalently, g has characteristic polynomial $(x - 1)^k$ for some $k \in \mathbb{N}$.

Here is the main theorem for the talk:

Theorem 1.0.2: Ratner's Orbit Closure Theorem

Let $G \leqslant \mathbf{SL}(\ell, \mathbb{R})$ be a semisimple Lie group and $\Gamma \leqslant G$ be a lattice. Suppose that $V \leqslant G$ is a subgroup of G generated by unipotent elements, and that $x \in G/\Gamma$. Then there is a closed subgroup $L \leqslant G$ such that the closure of Vx is Lx. Furthermore, L can be chosen so that

1. L contains the identity component of V;
2. L has only finitely many connected components;
3. There is an L-invariant, finite measure on Lx.

Remark 1.0.3

1. We may not eliminate the assumption that V is generated by unipotent elements.
2. If G is connected and has no compact factors, then it is generated by unipotent elements.

2 Application to Totally Geodesic Subspaces

In a compact hyperbolic manifold, the closure of a geodesic can be a fractal. It is a consequence of Ratner's Theorem that this pathology does not occur for higher dimensional subspaces.

Corollary 2.0.1: Closures of totally geodesic subspaces

Assume M is a compact, hyperbolic n-manifold and $\pi : \mathbb{H}^n \to M$ is a covering map. Furthermore, let $\iota : \mathbb{H}^m \hookrightarrow \mathbb{H}^n$ (with $m \le n$) be any geodesic embedding. Set $f : \mathbb{H}^m \to M$ by $f = \pi \circ \iota$. If $m \ge 2$, then the closure of the image, $\overline{f(\mathbb{H}^m)}$, is a (totally geodesic) immersed submanifold of M.

Proof

Following Witte-Morris' book, we will only prove that the closure is a submanifold, not that it is totally geodesic. Let $G = \mathbf{SO}(1,n) = \mathrm{Isom}^+(\mathbb{H}^n)$, $V = \mathbf{SO}(1,m)$, and $x \in \iota(\mathbb{H}^m) \subseteq \mathbb{H}^n$. Then $G \curvearrowright \mathbb{H}^n$, $M = \Gamma\backslash\mathbb{H}^n$ for some lattice $\Gamma \leqslant G$, and $\iota(\mathbb{H}^n) = Vx$.

From Ratner's Orbit-Closure Theorem, we know there is $L \leqslant G$, for which $\overline{\Gamma V} = \Gamma L$. So

$$\overline{f(\mathbb{H}^m)} = \overline{\pi(\iota(\mathbb{H}^m)} = \overline{\pi(Vx)} = \pi(Lx)$$

is an immersed submanifold of M. ■

One might be led to consider this kind of proof by considering a totally geodesic subspace $\iota(\mathbb{H}^m) \subseteq \mathbb{H}^n$ as the orbit of one point under a subgroup of the isometries of $\mathbb{H}^n$. Since the groups $\mathbf{SO}(1,n)$ are generated by unipotent elements, Ratner's theorem may be applied in any setting where it is a subgroup of some larger ambient Lie group.

Remark 2.0.2

1. The same conclusion holds (same proof) in more general symmetric spaces that have no compact factors, though the closure may not be totally geodesic if the ranks are not equal.
2. When ranks are the same, one proves the submanifold is totally geodesic by showing that the subgroup L in the above proof is invariant under the appropriate Cartan involution of G.

3 Application to Quadratic Forms

The basic question here is that of what was once known as the Oppenheim conjecture. The question is basically the following: fix a quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ (with $n \geq 3$), and suppose Q is indefinite (that is, not positive or negative definite) and is not a scalar multiple of an integral form (that is, a form with integer coefficients), and is not degenerate (meaning it admits a linear change of variables making it a form with fewer variables): is $Q(\mathbb{Z}^n)$ dense in $\mathbb{R}$?

Corollary 3.0.1: Margulis' Theorem on Values of Quadratic Forms

Let Q be a quadratic form in $n \geq 3$ variables, and assume Q is indefinite, not a scalar multiple of an integral form, and nondegenerate. Then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

Proof

For simplicity, assume $n = 3$. Let $G = \mathbf{SL}(3, \mathbb{R})$ and $H = \mathbf{SO}(Q)^\circ = \{h \in G \mid Q(hx) \equiv Q(x)\}^\circ$. Since we have assumed Q is indefinite, we have $H \cong \mathbf{SO}(1,2)^\circ \cong \mathbf{PSL}(2, \mathbb{R})$, so H is generated by unipotent elements. One can show the only connected subgroups containing H are H and G. Therefore, Ratner's theorem tells us that either:

$$HG_\mathbb{Z} \text{ is closed, and } G_\mathbb{Z} \cap H \text{ is a lattice in } H, \quad \text{or} \quad \overline{HG_\mathbb{Z}} = G.$$

However, if $H_\mathbb{Z} = H \cap G_\mathbb{Z}$ is a lattice in H, then the Borel Density Theorem implies H is defined over $\mathbb{Q}$, which implies that Q is a scalar multiple of an integral form. Therefore, we conclude that $\overline{HG_\mathbb{Z}} = G$, meaning $HG_\mathbb{Z}$ is dense in G, so $HG_\mathbb{Z}(1,0,0)$ is dense in $G(1,0,0)$, meaning $H\mathbb{Z}^3$ is dense in $\mathbb{R}^3$. Then by continuity of Q, we conclude $Q(H\mathbb{Z}^3)$ is dense in $Q(\mathbb{R}^3)$. We also know by definition of H that $Q(H\mathbb{Z}^3) = Q(\mathbb{Z}^3)$, and $Q(\mathbb{R}^3) = \mathbb{R}$. Therefore, $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$. ∎

4 Bonus if Time Remains

Another immediate application is the following:

Corollary 4.0.1: Products of Lattices

If Γ_1, Γ_2 are any two lattices in a simple group G, then either Γ_1, Γ_2 are commensurable, so the product $\Gamma_1\Gamma_2$ is discrete, or $\Gamma_1\Gamma_2$ is dense in G.

Finally, the theorem admits some rephrasings in more dynamical and measure-theoretic language. In particular, the following versions of the theorem are helpful for proving equidistribution results, which are generally stronger than density theorems.

Theorem 4.0.2: Ratner's Equidistribution Theorem

Let $\{u^t\}$ be any one-parameter subgroup of G, $x \in G/\Gamma$, and $c(t) = u^t x$, for $t \in [0, \infty)$. Then there is a connected, closed subgroup L of G such that:

1. There is a (unique) L-invariant probability measure μ on Lx;
2. The curve c is uniformly distributed in Lx, with respect to μ;
3. The closure of $\{c(t) \mid t \in [0, \infty)\}$ is Lx (so Lx is closed in G/Γ);
4. $\{u^t\} \subseteq L$.

And in yet another form:

Theorem 4.0.3: Ratner's Classification of Invariant Measures

Suppose that $V \leqslant G$ is generated by unipotent elements, and μ is any ergodic V-invariant probability measure on G/Γ. Then there is a unique closed subgroup L of G, and some $x \in G/\Gamma$, such that μ is the unique L-invariant probability measure on Lx. Furthermore, L can be chosen so that:

1. L has only finitely many connected components;
2. L contains the identity component of V;
3. Lx is closed in G/Γ.