Jr Topology - Teichmüller Space via Hyperbolic Geometry

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1 Hyperbolic Structures on Surfaces

We start with some reminders about basic Riemannian geometry and hyperbolic structures in particular. The basic question today is about understanding hyperbolic structures on surfaces. In general, the story we will tell today can be adapted to the setting of connected, oriented surfaces with boundary components and punctures, but we will focus on the case where the surface S has no boundary or punctures.

Definition 1.0.1: Hyperbolic metrics

Let M be a manifold. A Riemannian metric g on M is said to be *hyperbolic* if it complete, finite area, has geodesic boundary, and is of constant sectional curvature -1.

Suppose we have a fixed hyperbolic Riemannian manifold (X, g). A *hyperbolic structure* for M is a diffeomorphism

$$\phi : M \to X.$$

The map ϕ is referred to as the *marking*; the pair (X, ϕ) is called a *marked hyperbolic surface*.

The following is an immediate and important result.

Proposition 1.0.2

Suppose the Riemannian manifold (M, g) is hyperbolic. Then the universal cover, $(\tilde{M}, \tilde{g})$ is a simply connected hyperbolic Riemannian manifold, and there is an isometric identification

$$D : \tilde{M} \to \mathbb{H}^n,$$

where $\mathbb{H}^n$ is n-dimensional hyperbolic space.

Here are some important notes about the above.

Remark 1.0.3: Basic facts

- Note that hyperbolic structures immediately give rise to a hyperbolic metric on M, just by pulling back the metric on X by ϕ.

- Any closed curve in S (a hyperbolic surface) not homotopic into a neighborhood of a puncture is homotopic to a unique closed geodesic.
- By the previous, associated to any hyperbolic metric $\mathcal{X}$ is a length function $\ell_{\mathcal{X}}$ sending a free homotopy class of closed curves to the length of its geodesic representative.
- There is a bijective correspondence between conjugacy classes in $\pi_1(S)$ and oriented closed geodesic curves in S.
- By the uniformization theorem, up to isometry, there is a unique simply connected surface with hyperbolic metric, namely $\mathbb{H}^2$. This is the previous proposition.

Now here's a reminder of the fundamental link between a surface's curvature and its topology:

Theorem 1.0.4: Gauss-Bonnet

Suppose M is a compact two-dimensional Riemannian manifold with boundary ∂M. Let K be the Gaussian curvature of M and k_g be the geodesic curvature of ∂M. Then

$$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

The significance in the above theorem is that it interlinks the topology of a surface with its geometry. The Euler characteristic of a compact, connected, oriented surface of genus g, with b boundary components and k punctures is given by

$$\chi(S) = 2 - 2g + (b - k).$$

According to Gauss-Bonnet, it's interesting to ask about hyperbolic structures on surfaces with genus $g \geq 2$ (let's assume no boundary). In the case of the torus, when $g = 1$, we have 0 Euler characteristic, and it is instead reasonable to ask about the space of unit area flat metrics on it. There is an analogous version of that story which is simpler than what we will discuss today.

2 Definitions of Teichmüller Space

Classically, Teichmüller space is the space of Riemann surface structures on a surface S up to isotopy. But we will take a different approach: that of classifying hyperbolic structures on a surface S. By the Riemann uniformization theorem, these are equivalent definitions.

Two hyperbolic structures $\phi_i : S \to X_i$ $(i = 1, 2)$ on S are said to be homotopic if there is an isometry $I : X_1 \to X_2$ so that the maps $I \circ \phi_1$ is homotopic to ϕ_2; i.e the following diagram commutes up to homotopy:

$$\begin{tikzcd} & S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ X_1 \arrow[rr, "I"] & & X_2 \end{tikzcd}$$

Definition 2.0.1: Teichmüller Spaces

Let S be a compact surface with finitely many (perhaps zero) points removed from the interior, and assume $\chi(S) < 0$. We may then define the *Teichmüller space* of S, denoted $\mathcal{T}(S)$, in the following ways:

$$\begin{aligned}\mathcal{T}(S) &= \{\text{hyperbolic structures on } S\}/(\text{homotopy}) \\ &= \{\text{hyperbolic metrics on } S\}/\mathrm{Diff}_0(S)\end{aligned}$$

From this definition, there is a way to get a topology on $\mathcal{T}(S)$. Let $\mathcal{S}$ be the collection of simple closed curves on S. Consider the map $\mathcal{T}(S) \to \mathbb{R}^{\mathcal{S}}$ by $\mathcal{X} \mapsto (\ell_{\mathcal{X}}(\alpha))_{\alpha\in\mathcal{S}}$. Giving $\mathbb{R}^{\mathcal{S}}$ the product topology, $\mathcal{T}(S)$ inherits the subspace topology via this embedding. This embedding fits Teichmüller space into the very-infinite-dimensional space $\mathbb{R}^{\mathcal{S}}$. However, our next characterization enables us to see that there is really just a finite dimensional parameterization of this space.

Proposition 2.0.2: $\mathcal{T}(S)$ as a Representation space

There is a natural bijective correspondence

$$\mathcal{T}(S) \longleftrightarrow \{\rho : \pi_1(S) \to \mathbf{PSL}_2(\mathbb{R}) \mid \rho \text{ is discrete and faithful}\}/\,\mathbf{PGL}_2(\mathbb{R}).$$

Proof

For brevity, say $\mathbf{DF}(\pi_1(S), \mathbf{PSL}_2(\mathbb{R})) = \{\rho : \pi_1(S) \to \mathbf{PSL}_2(\mathbb{R}) \mid \rho \text{ is discrete and faithful}\}$.

Given $[(X, \phi)] \in \mathcal{T}(S)$, there is an isometric identification of $\tilde{X}$ with $\mathbb{H}^2$, so that we get an embedding $\pi_1(X) \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^2) = \mathbf{PSL}_2(\mathbb{R})$. The marking ϕ gives an identification $\pi_1(S) \cong \pi_1(X)$, hence a representation $\rho : \pi_1(S) \to \mathbf{PSL}_2(\mathbb{R})$. Since $\pi_1(X) \curvearrowright \tilde{X}$ properly discontinuously by isometries (via $\mathrm{deck}(\tilde{X})$), the representation ρ is discrete and faithful.

It remains to be shown that the choices made are invariant up to conjugacy. (In particular: the choice of representative $(X, \phi) \in [(X, \phi)]$, the isometric identification of $\tilde{X} \cong \mathbb{H}^2$, and the choice of identification of $\pi_1(X) \cong \mathrm{deck}(\tilde{X})$). I omit the proof but none of this is too hard.

For the other direction, take $\rho \in \mathbf{DF}(\pi_1(S), \mathbf{PSL}_2(\mathbb{R}))$, and we claim $\rho(\pi(S)) \curvearrowright \mathbb{H}^2$ is a covering space action. It is clearly properly discontinuous (since $\rho(\pi_1(S))$ is discrete). So one must only show the action is free (if it weren't this ends up violating discreteness). Then set $X = \mathbb{H}^2/\rho(\pi_1(S))$ and X is diffeomorphic to S. Any diffeomorphism $S \to X$ which induces the corresponding homomorphism $\rho_* : \pi_1(S) \to \pi_1(X)$ is the desired marking (and well-defined up to homotopy).

Finally, if you replace ρ by one of its $\mathbf{PGL}_2(\mathbb{R})$ conjugates, we get a surface X' isometric to X and thus ρ and its conjugate correspond to the same point in $\mathcal{T}(S)$. ∎

The proposition naturally gives $\mathcal{T}(S)$ a topology as well, as a subspace of $\mathrm{Hom}(\pi_1(S), \mathbf{PSL}_2(\mathbb{R}))$. In particular, give $\pi_1(S)$ the discrete topology, $\mathbf{PSL}_2(\mathbb{R})$ its usual topology as a Lie group, and

$\mathrm{Hom}(\pi_1(S), \mathbf{PSL}_2(\mathbb{R}))$ the compact-open topology. There is a way to describe this topology more concretely as inheriting the subspace topology from the Lie group $\mathbf{PSL}_2(\mathbb{R})^{2g}$, since any homomorphism $\pi_1(S) \to \mathbf{PSL}_2(\mathbb{R})$ is determined by where it sends the $2g$ generators of $\pi_1(S)$ (different choices of generating sets yield equivalent topologies). These are equivalent constructions.

3 Dimension Count

3.1 Via representation spaces

We use the characterization of $\mathcal{T}(S)$ as a space of discrete and faithful representations of $\pi_1(S)$. Note that $\pi_1(S)$ has the following presentation:

$$\pi_1(S) = \langle \gamma_1, \gamma_2, ..., \gamma_{2g} \mid [\gamma_1, \gamma_2]...[\gamma_{2g-1}, \gamma_{2g}] = 1 \rangle$$

First, the group $\mathbf{PGL}_2(\mathbb{R})$ is 3-dimensional and acts on DF with 3-dimensional orbits, so $\dim(\mathbf{DF}(\pi_1(S), \mathbf{PSL}_2(\mathbb{R}))/\,\mathbf{PGL}_2(\mathbb{R})) = \dim(\mathbf{DF}(\pi_1(S), \mathbf{PSL}_2(\mathbb{R}))) - 3$.

A representation $\rho \in \mathbf{DF}$ is determined by choosing the $2g$ images $\rho(\gamma_i)$. However, note that $\rho(\gamma_{2g})$ is determined by the other images due to the 1 relation. So we have $3(2g-1) = 6g-3$ choices to make. Therefore, $\dim \mathcal{T}(S) = 6g-6$.

3.2 Via tiling spaces

A hyperbolic S_g tile is a $4g$-gon in hyperbolic plane satisfying:

1. Sum of interior angles is 2π
2. Reading clockwise, edges are labeled $\gamma_1, \gamma_2, \gamma_1, \gamma_2, ..., \gamma_{2g-1}, \gamma_{2g}, \gamma_{2g-1}, \gamma_{2g}$.
3. Edges with the same labels have the same hyperbolic length

It turns out that $\mathcal{T}(S_g)$ can be identified with the space of equivalence classes of hyperbolic S_g tiles (these are fundamental domains in $\mathbb{H}^2$ corresponding to a discrete subgroup $\Gamma \leqslant \mathrm{Isom}^+(\mathbb{H}^2)$, whose quotient under side identifications is the surface S_g).

Counting the dimension goes as follows:

$+\,8g$ choosing the 4g vertices
$-\,2g$ side lengths matching in pairs
$-\,1$ scale so interior angles sum to 2π
$-\,3$ isometric tilings are equivalent
$-\,2$ pushing base point gives different tilings representing the same point in Teichmüller space
$=6g-6$ total dimensions.

4 Pairs of pants and Fenchel-Nielsen Coordinates

To formally prove that Teichmüller space is a ball of dimension $6g-6$, we consider pants decompositions of the surface S. This is equivalent to choosing $3g-3$ oriented simple closed curves, which determine a decomposition of the surface into $2g-2$ pairs of pants.

Length parameters: given an isotopy class c of simple closed curves in S, define the map $\mathcal{T}(S) \to \mathbb{R}$ by $\mathcal{X} \mapsto \ell_{\mathcal{X}}(c)$ (that is, the length of the geodesic representative of the isotopy class); this map is continuous.

The collection of hyperbolic structures on pairs of pants is determined by the length parameters of their boundaries.

Proposition 4.0.1

Suppose P is a pair of pants with boundary components $\alpha_1, \alpha_2, \alpha_3$. The map $\mathcal{T}(P) \to \mathbb{R}^3_+$ defined by

$$\mathcal{X} \mapsto (\ell_{\mathcal{X}}(\alpha_1), \ell_{\mathcal{X}}(\alpha_2), \ell_{\mathcal{X}}(\alpha_3))$$

is a homeomorphism.

So the hyperbolic structure on a pair of pants is determined by the length parameters for its boundary components, but for this to determine a hyperbolic metric on S, we also need to specify how the pants glue together. This is done via the *twist parameters*.

(Draw picture here)

Also leads to a natural flow on Teichmüller space (the twist flow).

Theorem 4.0.2

Let S_g be a closed, connected, oriented surface of genus $g \geq 2$. Then

$$\mathcal{T}(S_g) \cong \mathbb{R}^{6g-6}.$$

5 Higher Teichmüller Theory

A natural generalization of one our aforementioned characterizations of Teichmüller space is to consider more generally

$$\mathbf{DF}(\pi_1(S), G) = \{\rho : \pi_1(S) \to G \mid \rho \text{ is discrete and faithful}\},$$

where G is some fixed semisimple Lie group. We consider such representations up to conjugacy in G.

The basic questions are to understand the structure of this space and the geometric significance behind these representations. In particular, studying these representations is a very different story in higher rank Lie groups and it is an active area of research.