# Jr Topology: Thurston Geometrization

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## 1 Homogeneous Geometry in Dimension Two

The story of Euclid's axioms for geometry and the infamous parallel postulate.

Three cases: $\mathbb{R}^2, S^2, \mathbb{H}^2$. Equivalent conditions in terms of parallel lines, sums of angles in triangles, etc.

**Definition 1.0.1: Homogeneous manifolds**

A Riemannian manifold $(M, g)$ is called **homogeneous** if the group of isometries, $\text{Isom}(M, g)$ acts transitively on $M$. This means that given $p, q \in M$, there is some $\phi \in \text{Isom}(M, g)$ for which $\phi(p) = q$.

The aforementioned examples of model spaces are readily seen to be homogeneous. As a consequence of homogeneity (and being two-dimensional), the curvature at each point of these model spaces is identical. The following theorem governs the possibilities for trying to equip a 2-manifold with geometry like this, with constant curvature.

**Theorem 1.0.2: Uniformization for Surfaces**

Every compact, connected, smooth 2-manifold admits a locally frame-homogeneous Riemannian metric and a Riemannian covering by $\mathbb{R}^2, \mathbb{H}^2$, or $S^2$ (equipped with their usual metrics).

The proof of the above relies on the classification of surfaces. Then, one can detect which kind of metric the space can be equipped with purely topologically via the Gauss-Bonnet theorem.

**Theorem 1.0.3: Gauss-Bonnet**

If $(M, g)$ is a smoothly triangulated compact Riemannian 2-manifold, then

$$\int_M K \, dA = 2\pi\chi(M),$$

where $K$ is the Gaussian curvature of $g$ and $dA$ is its Riemannian density.

In other words, the sign of average curvature must agree with the sign of the Euler characteristic, which is a topological invariant.

## 2 Moving to Dimension Three

In dimension two, the study is considerably simplified by the classification theorem for compact, connected surfaces.

**Theorem 2.0.1: Classification of Surfaces**

Every compact, connected, orientable 2-manifold is homeomorphic to a sphere or a connected sum of $n$ tori. Every nonorientable compact, connected 2-manifold is homeomorphic to a connected sum of $n$ copies of the real projective plane $\mathbb{R}P^2$.

In dimensions 3 and higher, there is no simple analogue of this theorem. Thurston's geometrization is an attempt to attack this classification problem via a geometric angle.

## 3 The Eight Model Geometries

There are eight homogeneous, simply connected, complete Riemannian 3-manifolds.

Three homogeneous geometries in dimension 3 are realized as direct generalizations of their 2-dimensional counterparts:

$$\mathbb{R}^3, \mathbb{H}^3, S^3$$

These first three are isotropic and have constant sectional curvature, while the others we will see have neither of these properties. Another two arise from twisting together two lower-dimensional examples:

$$\mathbb{H}^2 \times \mathbb{R}, S^2 \times \mathbb{R}$$

The last three examples come from Lie groups:

$$\mathrm{Nil}, \mathrm{Sol}, \mathbf{S\tilde{L}}_2$$

We describe some relevant features below:

- Nil is an $\mathbb{R}$-bundle over $\mathbb{R}^2$, whose geometry is fully carried by the Heisenberg group of matrices of the form
$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$
This collection is called Nil because it is a non-abelian nilpotent Lie group. There is a Lie groups exact sequence
$$0 \to \mathbb{R} \to \mathrm{Nil} \to \mathbb{R}^2 \to 0$$
where the first $\mathbb{R}$ is the center of Nil. So fix the regular Euclidean inner product on $T_e\mathrm{Nil} = \mathbb{R}^3$ and then translate this metric around by left multiplication. This gives Nil geometry.

- $\widetilde{\mathbf{SL}}_2$ geometry is similar to Nil but it's an $\mathbb{R}$ bundle over $\mathbb{H}^2$. Basically one takes the Riemannian structure induced on the unit tangent bundle of $\mathbb{H}^2$, where we have $T^1\mathbb{H}^2 \cong \mathbf{PSL}_2(\mathbb{R})$. Then, using the covering sequence

$$\widetilde{\mathbf{SL}}_2 \to \mathbf{SL}_2(\mathbb{R}) \to \mathbf{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$$

one can pull back the Riemannian metric to $\widetilde{\mathbf{SL}}_2$.

- Sol is an $\mathbb{R}^2$-bundle over $\mathbb{R}$. The geometry is governed by the Lie group Sol which is $\mathbb{R}^3$ with the operation

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z}x', y + e^z y', z + z'),$$

and note that the subgroup of all elements $(x, y, 0)$ is the center of Sol, so there is an exact sequence

$$0 \to \mathbb{R}^2 \to \mathrm{Sol} \to \mathbb{R} \to 0$$

hence the bundle statement. Sol is solvable but not nilpotent. At the point $(x, y, z)$, assign the scalar product

$$\begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It turns out a smooth 3-manifold can be determined to admit a **geometric structure** modeled on one of the eight aforementioned geometries purely in terms of features of its fundamental group. By this, we mean the following:

**Definition 3.0.1**

Let $M$ be one of the aforementioned eight special 3-manifold geometries. A Riemannian 3-manifold $N$ has a **geometric structure modeled on** $M$ if $N$ is locally isometric to $M$; that is, for any $p \in N$, there is a neighborhood $p \in U_p$ where $U_p$ is isometric to some open set in $M$.

If $N$ has a geometric structure modeled on $M$, then $N$ is locally homogeneous: any two points $p, q \in N$ admit isometric neighborhoods $U_p \cong U_q$, each of which is isometric to an $\varepsilon$-ball at any point of $M$.

The following gives an important reason to study when a given manifold admits a certain kind of geometry:

**Theorem 3.0.2**

Two closed 3-manifolds admitting different geometries are not diffeomorphic, and not even commensurable (admitting a common cover with finite degrees).

The following theorem says one can determine the type of geometric structure a manifold admits purely in terms of its fundamental group:

**Proposition 3.0.3**

Let $M$ be a closed manifold modeled on one $\mathbb{X}$ of the eight geometries. Then:

1. if $\pi_1(M)$ is finite, then $\mathbb{X} = S^3$, otherwise;
2. if $\pi_1(M)$ is virtually cyclic, then $\mathbb{X} = S^2 \times \mathbb{R}$, otherwise;
3. if $\pi_1(M)$ is virtually abelian, then $\mathbb{X} = \mathbb{R}^3$, otherwise;
4. if $\pi_1(M)$ is virtually nilpotent, then $\mathbb{X} = \text{Nil}$, otherwise;
5. if $\pi_1(M)$ is virtually solvable, then $\mathbb{X} = \text{Sol}$, otherwise;
6. if $\pi_1(M)$ contains a normal cyclic subgroup $K$, then:
   - if a finite index subgroup of the quotient lifts, $\mathbb{X} = \mathbb{H}^2 \times \mathbb{R}$,
   - otherwise $\mathbb{X} = \tilde{\mathbf{SL}}_2$
7. otherwise $\mathbb{X} = \mathbb{H}^3$.

# 4 Thurston's Geometrization

Here is the statement of Thurston's geometrization theorem:

**Theorem 4.0.1: Geometrization**

Let $M$ be an irreducible orientable compact 3-manifold with (maybe empty) boundary consisting of tori. Every block of the geometric decomposition of $M$ is geometric.

This theorem was conjectured in 1982 by Thurston and eventually proven in 2002 by Perelman. His proof involved detailed geometric analysis of the Ricci flow.

# 5 Corollaries

**Theorem 5.0.1: Poincare**

Every closed simply connected 3-manifold $M$ is diffeomorphic to $S^3$.

**Theorem 5.0.2**

Let $\tilde{M} \to M$ be a finite covering. If $\tilde{M}$ is geometric, then so is $M$ (with the same geometry).

**Theorem 5.0.3**

Every simple compact manifold $M$ bounded by a non-empty collection of tori is hyperbolic.