Jr Topology: Nielsen Realization

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1 The Basic Topology of Surfaces

First, let's recall some basic facts about the topology of surfaces. Closed or compact surfaces are very nicely classified.

Theorem 1.1: Classification of Surfaces

Suppose S is a closed, connected, orientable surface. Then S is homeomorphic to the sphere S^2, or $S^2 \#_g \mathbb{T}^2$ for some $g \in \mathbb{N}$, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the torus and $\#_g$ denotes the operation of g-fold connect sum. The number g is called the *genus* of the surface S.

Suppose instead S is compact, connected oriented surface. Then S is obtained by taking a closed, connected, oriented surface, and removing $b \in \mathbb{N}$ disjoint open disks (with disjoint closures) from it. These removed disks leave behind b *boundary components*.

From the above, it is clear that there is a bijective correspondence between homeomorphism types of compact, connected, oriented surfaces and the set $\{(g, b) \mid g, b \geq 0\}$.

One may obtain noncompact surfaces by taking a compact, connected, oriented surface and removing n points from it, called *punctures*.

Let $S_{g,b,n}$ denote the unique surface of genus g, with b boundary components, and n punctures.

Lemma 1.2: Euler Characteristics of Surfaces

$$\chi(S_{g,b,n}) = 2 - 2g - (b + n)$$

This is an important observation, because the following theorem informs the kind of geometry that can be put on surfaces.

Theorem 1.3: Gauss-Bonnet

Suppose (S, g) is a compact, connected, oriented, complete, 2-dimensional Riemannian manifold. Then

$$\int_S K\, dA + \int_{\partial S} k\, ds = 2\pi\chi(S),$$

where K is the Gaussian curvature of the metric g and k is the geodesic curvature of ∂S.

This theorem means that if we put a Riemannian metric on a surface with negative Euler characteristic, on average, the metric must be negatively curved. It turns out that hyperbolic metrics always exist in this case (a consequence of the Riemann Uniformization Theorem).

Theorem 1.4: Hyperbolic Geometry on Surfaces

Suppose S is a compact, connected, orientable surface. If $\chi(S) < 0$, then S admits a hyperbolic metric. If $\chi(S) = 0$, then S admits a Euclidean metric.

We will now briefly review hyperbolic geometry and its importance below.

2 Hyperbolic Geometry and Teichmüller Space

Stuff about the hyperbolic plane and its group of isometries.

Upper half plane, $\mathbf{PSL}_2(\mathbb{R})$ as the group of isometries.

Definition 2.1: Teichmüller Space

Define the *Teichmüller Space of* S as

$$\mathcal{T}(S) = \mathrm{HypMet}(S)/\mathrm{Diff}_0(S),$$

in other words the space of hyperbolic metrics on the space considered up to pulling back under diffeomorphisms isotopic to the identity.

3 More Facts about Surfaces

Here is a brief collection of some important facts about self-homeomorphisms of S.

Theorem 3.1: Homotopic Homeos are usually Isotopic

Let S be any compact surface, and f, g two orientation-preserving homotopic homeomorphisms of S. Then f, g are isotopic. This holds true even for homotopy and isotopy relative to ∂S.

Theorem 3.2

Let S be a compact surface. Then every homeomorphism of S is isotopic to a diffeomorphism of S.

These theorems essentially allow one to freely pass between the notions of homotopy and isotopy, as well as homeomorphism and diffeomorphism in discussing mapping class groups, as we will see below. This is especially convenient at times.

4 Mapping Class Groups

One might hope to study surfaces, or more generally any kind of topological space, by considering the collection of self-homeomorphisms of the space. This collection is naturally a group under composition. The only issue is that most of the time, this group is unfathomably large.

There are several approaches to restricting this group which might yield topological insight. The main player for us is when we consider self-homeomorphisms up to isotopy.

Definition 4.1: Mapping Class Groups

Suppose S is a surface (with or without boundary). Let $\mathrm{Homeo}^+(S, \partial S)$ denote the group of orientation-preserving homeomorphisms which fix ∂S pointwise.
The *mapping class group* of S is the group

$$\begin{aligned}
\mathrm{MCG}(S) = \mathrm{Mod}(S) &= \pi_0(\mathrm{Homeo}^+(S, \partial S)) \\
&\cong \mathrm{Homeo}^+(S, \partial S)/\text{homotopy} \\
&\cong \pi_0(\mathrm{Diff}^+(S, \partial S)) \\
&\cong \mathrm{Diff}^+(S, \partial S)/\sim
\end{aligned}$$

where $\sim$ denotes either smooth homotopy or smooth isotopy relative to the boundary.

Important: mapping class group treats boundaries and punctures differently. May permute punctures.

5 Examples and Computations

Mapping class groups are not always trivial: order g (rotation) homeomorphism of S_g. Not isotopic to the identity because there are simple closed curves with pairwise non-isotopic orbits under application of this map.

Most elements of mapping class groups are of infinite order (Dehn twists, for example).

Here is a list of examples of mapping class groups:

- $\mathrm{MCG}(D^2) \cong \mathrm{MCG}(S^2) \cong \mathrm{MCG}(S_{0,1}) \cong \{e\}$; any two homeomorphisms fixing the boundary are isotopic via the straight line homotopy (not D^2, but still explicit)
- $\mathrm{MCG}(S_{0,3}) \cong \Sigma_3$ and $\mathrm{MCG}(S_{0,2}) \cong \mathbb{Z}/2\mathbb{Z}$; permuting the punctures differentiates isotopy classes of homeomorphisms
- Let A be the annulus, then $\mathrm{MCG}(A) \cong \mathbb{Z}$; curves wrap around
- $\mathrm{MCG}(\mathbb{T}^2) \cong \mathbf{SL}_2(\mathbb{Z})$; toral automorphisms not isotopic
- $\mathrm{MCG}(S_{0,4}) \cong \mathbf{PSL}_2(\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$

6 Finite Order

Suppose we are given a finite order mapping class $[\phi]$, let's say of order k. By definition, $[\phi]^k = [\phi^k] = [\mathrm{Id}]$, which means only that $\phi^k \sim \mathrm{Id}$, not that ϕ^k is itself the identity. This begs the question: can we identify $\psi \in [\phi]$ such that in fact we have $\psi^k = \mathrm{Id}$?

The answer is yes, and this was known to Nielsen in the 30s. Furthermore, the element ψ can be chosen to be an isometry for some hyperbolic structure on S.

In general, the collection of isometries for a given metric is always finite.

Theorem 6.1

Let X be a hyperbolic surface homeomorphic to S_g with $g \geq 2$. Then $\mathrm{Isom}(X)$ is a finite group. In fact, $|\mathrm{Isom}^+(X)| \leq 84(g-1)$.

Also, no nontrivial isometry of X is isotopic to the identity.

7 The Nielsen Realization Problem

The question in the previous section can be upgraded. Given an arbitrary finite subgroup of $\mathrm{MCG}(S)$, is it always realized as a subgroup of isometries for some hyperbolic metric on S?

Theorem 7.1: The Nielsen Realization Theorem (Kerckhoff '83)

Let $S = S_{g,n}$ have $\chi(S) < 0$. If $G \leqslant \mathrm{MCG}(S)$ is finite, then there exists a finite $\tilde{G} \leqslant \mathrm{Homeo}^+(S)$ so that the projection $\mathrm{Homeo}^+(S) \to \mathrm{MCG}(S)$ restricts to an isomorphism $\tilde{G} \to G$. Furthermore, $\tilde{G}$ can be chosen to be a subgroup of isometries for some fixed hyperbolic metric on S.

8 Methods and Concepts in the Proof

The action of the mapping class group on Teichmuller space (Fricke's theorem); fixing points in Teichmuller space

Theorem 8.1: Fricke's Theorem

If S is a closed, connected, oriented surface of genus $g \geq 2$, then $\mathrm{MCG}(S)$ acts properly discontinuously on $\mathcal{T}(S)$.

The key idea is to try to find fixed points for this action; if one can show there is a fixed point for any finite subgroup, then one will have proven the Nielsen Realization Theorem.