Jr Topology: Geodesic Currents, Laminations, and Teichmüller Space

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April 2024

1 General background

Throughout the talk we will be imagining we have fixed a compact, orientable surface S of genus $g \geq 2$.

Recall that for any manifold M, $\pi_1(M) \curvearrowright \tilde{M}$ by deck transformations.

Here we will discuss a couple different perspectives for hyperbolic structures on manifolds, or more specifically surfaces.

1.1 ■ Riemannian geometry perspective

Definition A Riemannian metric g on S is a smoothly varying inner product on each tangent space T_pS. Formally, for each $p \in S$, there is an inner product $g_p : T_pS \times T_pS \to \mathbb{R}$.

Any Riemannian structure gives the surface S a notion of curvature. A Riemannian metric is said to be hyperbolic if it is of constant curvature -1.

Riemannian structures enable one to talk about lengths of curves, and they give S the structure of a metric space (infimum of lengths of curves between two points). A geodesic is a (locally) length-minimizing path. The following is an important theorem in Riemannian geometry.

Theorem (Cartan) Every closed curve in a Riemannian manifold (S, g) is freely homotopic to a unique closed geodesic.

In other words, there is a unique geodesic in every free homotopy class of simple closed curves. A useful perspective for us: fix the surface S, then any choice of hyperbolic Riemannian metric g for S gives a unique geodesic representative of every element in $\pi_1(S)$.

1.2 ■ (G, X) structure perspective

A hyperbolic structure on S can also be described via a so-called developing map $D : \tilde{S} \to \mathbb{H}^2$, which comes equipped with a holonomy representation $\rho : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$. This yields an identification $S \cong \mathbb{H}^2/\rho(\pi_1(S))$, and S inherits a hyperbolic structure this way.

The perspective we take throughout the talk is that the choice of a hyperbolic metric m on S gives an isometric diffeomorphism $\varphi_m : \tilde{S} \to \mathbb{H}^2$, which is well defined up to conjugation by an isometry of $\mathbb{H}^2$.

2 Teichmüller space

Given a compact orientable surface S of genus $g \geq 2$, the Teichmüller space $\mathcal{T}(S)$ is the space of isotopy classes of all Riemannian structures on S of constant curvature -1. $\mathcal{T}(S)$ can also be described as the representation space of injective representations (up to conjugation, with discrete image) from $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ Here are just a couple of important facts about Teichmüller space.

Facts Let S_g be a surface of genus g. Then

1. There is a homeomorphism $\mathcal{T}(S_g) \cong \mathbb{R}^{6g-6}$
2. $\mathcal{T}(S_g)$ comes equipped with a natural Riemannian metric called the Weil-Petersson metric

3 Geodesic and measured laminations

A geodesic lamination λ is a closed subset of S consisting of a union of geodesics. Think of this as a partial foliation of S by geodesics (a closed subset consisting of a union of disjoint simple geodesics). The "leaves" of the lamination are the particular geodesics which make it up.

A measured geodesic lamination consists of a geodesic lamination equipped with a transverse measure, that is a measure defined on each arc k transverse to λ, with support $k \cap \lambda$, and which is invariant by homotopy respecting λ.

Theorem (Thurston) Every geodesic lamination admits a transverse measure, possibly with smaller support.

Thurston introduced this notion to define a completion of the space of simple closed geodesics. By talking about a space of measures, this provides a way to talk about convergence separate from other geometric means one would need to take if only working with the surface.

4 The space of geodesic currents

Notice that a choice of hyperbolic structure on the surface S gives an identification $\tilde{S} \cong \mathbb{H}^2$. We can describe the collection of (undirected) geodesics on $\mathbb{H}^2$ by $G(\mathbb{H}^2) = (\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \Delta)/\mathbb{Z}_2 = (S^1 \times S^1 \setminus \Delta)/\mathbb{Z}_2$ (a möbius band). A priori, we must fix a geometric structure on S in order to obtain an explicit description of the set $G(\tilde{S})$, so it's not necessarily clear that this doesn't depend on the geometry. But indeed, the topology of $G(\tilde{S})$ can be abstractly described in terms of the group $\pi_1(S)$ only, and it doesn't depend on a choice of hyperbolic metric. If I want to discuss the circle at infinity:

Lemma Suppose M is a compact manifold. Then $\pi_1(M)$ is finitely generated. (Even finitely presented.)

Otherwise skip, then:

Definition Fix a surface S of genus $g \geq 2$. The space of *geodesic currents*, denoted $\mathcal{C}(S)$, is the collection of (locally finite, nonzero) $\pi_1(S)$-invariant measures on $G(\tilde{S})$.

Note that $\mathcal{C}(S)$ comes endowed naturally with the weak-$*$ topology (as it is a space of measures). In particular, this means the measures $(\mu_n) \to \mu$ if and only if $\int_{G(\mathbb{H}^2)} f d\mu_n \to \int_{G(\mathbb{H}^2)} f d\mu$ for every compactly supported $f : G(\mathbb{H}^2) \to \mathbb{R}$. Additionally, it makes sense to add and scale geodesic currents, so it has the structure of a cone.

We can construct geodesic currents by taking the lift of any closed geodesic to the universal cover, and equipping it with delta masses on each lift. Therefore, we have the following theorem.

Theorem There is an inclusion from the real multiples of homotopy classes of closed curves to the space $\mathcal{C}(S)$. Furthermore, this inclusion is dense in the uniform space $\mathcal{C}(S)$.

Remark: the second part of that statement is nontrivial.

5 Embedding of Teichmüller space

Consider the space of all marked hyperbolic structures on S, denoted by $\mathcal{T}(S)$. We will see there is a natural embedding $\mathcal{T}(S) \hookrightarrow \mathcal{C}(S)$. This embedding comes from the *Liouville measure* associated to a choice of hyperbolic structure on S.

Definition The Liouville measure L on $G(\mathbb{H}^2)$ is given by $L([a,b]\times[c,d]) = \left| \log\left(\frac{(a-c)(b-d)}{(a-d)(b-c)}\right) \right|$.

Note that this measure is invariant under $\mathrm{PSL}(2,\mathbb{R})$, since cross ratios are preserved under this action. A relatively straightforward calculation shows that this measure is absolutely continuous with respect to the measure on $G(\mathbb{H}^2)$ induced by Lebesgue measure on S^1.

Now fix a hyperbolic structure m on S, which gives an isometric diffeomorphism $\varphi_m : \tilde{S} \to \mathbb{H}^2$, and this specifies a measure L_m on $G(\tilde{S})$ (the Liouville measure inherited by the identification $\tilde{S} \cong \mathbb{H}^2$). This measure is clearly invariant under $\pi_1(S)$ due to Liouville measure being invariant under the isometries of $\mathbb{H}^2$. Also, changing the choice of φ_m by conjugation with an isometry of $\mathbb{H}^2$ does not change the underlying measure.

Theorem (Bonahon)

1. If m, m' are isotopic hyperbolic structures on S, then the Liouville measures $L_m = L_{m'}$
2. If $m \neq m'$ in $\mathcal{T}(S)$, then L_m and $L_{m'}$ are not absolutely continuous with respect to

one another.

3. The map $m \mapsto L_m$ is continuous (hyperbolic structures have the C^∞ topology)

4. The geodesic current L_m depends only on the isotopy class of m

5. There is a continuous, injective, proper map $\mathcal{T}(S) \to \mathcal{C}(S)$ by $[m] \mapsto L_m$

6. The previous map is actually a homeomorphism onto its image

Similarly, we will find an embedding $\pi_1(S) \hookrightarrow \mathcal{C}(S)$. More generally, there is an embedding from the space of measured geodesic laminations $\mathcal{ML}(S) \hookrightarrow \mathcal{C}(S)$.

6 Intersection numbers of geodesic currents

For two homotopy classes of closed curves α, β, there is a natural notion of *geometric intersection number*, defined to be the infimum of the cardinality of $\alpha' \cap \beta'$ for all $\alpha' \in \alpha, \beta' \in \beta$. A natural question is whether this notion extends to the space of geodesic currents in a natural way. Indeed, there is such a notion.

Definition Given two $\alpha, \beta \in \mathcal{C}(S)$, define their *intersection number* to be $i(\alpha, \beta) =$ the total mass of the product measure $\alpha \times \beta$ on the set $DG(S) = DG(\tilde{S})/\pi_1(S)$, where $DG(\tilde{S}) \subseteq G(\tilde{S}) \times G(\tilde{S})$ is the subset of all pairs of geodesics in the universal cover which have a transverse non-trivial intersection.

Note that there is an identification $DG(\tilde{S})$ with the collection of triples $(\tilde{x}, \lambda_1, \lambda_2)$, where λ_1, λ_2 are distinct directions in the tangent space at $\tilde{x} \in \tilde{S}$. Since $\pi_1(S)$ acts properly discontinuously on $\tilde{S}$, it makes sense to take this quotient; this stands in contrast to taking the quotient of $G(\tilde{S})$ by the action of $\pi_1(S)$.

This intersection number allows for a number of interesting constructions, including the projectivization of the space of geodesic currents, the identification of which currents arise from measured laminations, and a reltionship with the celebrated Weil-Petersson metric on Teichmüller space.

Properties of i

1. $i : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}^+$ is symmetric, bilinear, and continuous*.

2. The intersection number of two geodesic currents arising from homotopy classes of closed curves is equal to their geometric intersection number.

3. Given α a closed curve, $m \in \mathcal{T}(S)$, $i(\alpha, L_m) = l_m(\alpha)$, the length of α in the metric m.

4. For all $m \in \mathcal{T}(S)$, $i(L_m, L_m) = \pi^2|\chi(S)|$ (note it is independent of m).

5. The set of $\alpha \in \mathcal{C}(S)$ for which $i(\alpha, \alpha) = 0$ is naturally homeomorphically identified with the space $\mathcal{ML}(S)$ of measured geodesic laminations on S.

There is a geodesic current α for which: every geodesic of $\tilde{S}$ transversely intersects another geodesic $\in \operatorname{supp}(\alpha)$ in $G(\tilde{S})$.

Proposition Fix α with the above property. Then the set $\{\beta \in \mathcal{C}(S) \mid i(\alpha, \beta) \leq 1\}$ is compact. This implies that the space of projective geodesic currents, $\mathcal{PC}(S) = \mathcal{C}(S)/\mathbb{R}^+$ is compact (realize the quotient by sending $\beta \mapsto \beta/i(\alpha, \beta)$, and the image is compact).

Consider the set of formal differences $\mathcal{C}_\pm(S)$ of elements of $\mathcal{C}(S)$. The intersection number extends to this space in a way that gives the structure of a metric on the subspace $\mathcal{T}(S) \subseteq \mathcal{C}_\pm$. This metric is actually equivalent to the Weil-Petersson metric up to multiplication by a constant.