Jr Geometry - Geodesic Flows and Ergodicity in Negative Curvature

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1 Basic Riemannian Geometry

We begin with some basics about the geometric setting we will find ourselves in today.

Definition 1.0.1: Riemannian Manifold

Let M be an n-dimensional manifold. A *Riemannian metric* g on M is a smoothly varying choice of positive definite inner product on each tangent space.

Riemannian metrics bring the notions of length, distance, angles, and more to the setting of abstract manifolds. In particular, we have the following:

Remark 1.0.2: Facts about Riemannian manifolds

Suppose (M, g) is a Riemannian manifold. Then:

- The length of a smooth curve $\gamma : I \to M$ is given by $\ell(\gamma) = \int_I \|\dot{\gamma}(t)\|_g dt = \int_I \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_g} dt$;
- (M, g) is a metric space with distance between two points $p, q \in M$ is defined by $d(p, q) = \inf\{\ell(\gamma) \mid \gamma : [a, b] \to M, \gamma(a) = p, \gamma(b) = q\}$;
- If $p \in M$, $u, v \in T_pM$, then $\theta = \arccos\left(\frac{g_p(u,v)}{\|u\|_g \|v\|_g}\right)$ is the angle between u, v.

The analogue of a straight line in Euclidean space is what's called a geodesic. These are locally length-minimizing paths.

Definition 1.0.3: Geodesic

Let $p, q \in M$. Suppose there is a path $\gamma : [a, b] \to M$ with $\gamma(a) = p, \gamma(b) = q$ for which $d(p, q) = \ell(\gamma)$. Then γ is called a *geodesic*.
This definition is not quite the proper one but it will suffice for our purposes.

Theorem 1.0.4: Hopf-Rinow

A Riemannian manifold M is *geodesically complete*, meaning every geodesic can be extended indefinitely, if and only if M is complete as a metric space. This is also equivalent to asserting that every closed and bounded set in M is compact.

Remark 1.0.5

Hyperbolic space is geodesically complete (and therefore so are any compact quotients of it by subgroups of its isometry group).

Now, we introduce a natural space which parameterizes the geodesics on hyperbolic surfaces.

Definition 1.0.6: The Unit Tangent Bundle

Suppose (M, g) is a Riemannian manifold. Define the *unit tangent bundle* of M to be the collection

$$T^1M = SM = \{(p, v) \in TM \mid |v|_g = 1\}.$$

This represents the collection of points in M along with all possible directions of travel from those points. It is a *sphere bundle* over M; if $\pi : SM \to M$ denotes the usual projection by $\pi(p, v) = p$, then for all $p \in M$ we have that $\pi^{-1}(p) \cong S^{n-1}$.

This is the natural space on which we will define the geodesic flow.

2 The Geodesic Flow

The intuitive idea here is to take a point with a unit tangent vector and flow it at unit speed along geodesics, transporting the unit vector along the way by keeping it tangent to the curve.

Definition 2.0.1: Geodesic Flow on the Unit Tangent Bundle

Suppose (M, g) is a complete Riemannian manifold and SM is its unit tangent bundle. If $p \in M$, $v \in SM_p$ and $\gamma : \mathbb{R} \to M$ is a unit-speed geodesic for which $\gamma(0) = p, \dot{\gamma}(0) = v$, then define the flow $\phi^t : SM \to SM$ by $(p, v) = (\gamma(0), \dot{\gamma}(0)) \mapsto (\gamma(t), \dot{\gamma}(t))$.

Example

1. Consider the geodesic flow on S^n: all orbits are periodic
2. In $\mathbb{R}^n$ and $\mathbb{H}^n$: no recurrence
3. In $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$: some recurrent orbits (at rational slopes), some orbits don't
4. In $\mathbb{H}^n/\Gamma$: chaos

3 Anosov flows

We will be able to easily see that geodesic flows on negatively curved manifolds are Anosov flows. Here is the definition.

Definition 3.0.1: Anosov flow

A smooth flow $\phi^t : M \to M$ (where M is a compact Riemannian manifold) is called *Anosov* if it has no fixed points and there are distributions $E^s, E^u \subseteq TM$ and constants $C \in \mathbb{R}^+, \lambda \in (0,1)$ such that for any $x \in M, t \geq 0$, we have

1. $E^s(x) \oplus E^u(x) \oplus E^o(x) = T_x M$, where E^o is the line bundle tangent to the flow;
2. $||d\phi^t_x v_s|| \leq C\lambda^t \|v_s\|$ for any $v_s \in E^s(x)$; and
3. $||d\phi^{-t}_x v_u|| \leq C\lambda^t \|v_u\|$ for any $v_u \in E^u(x)$.

In other words, there is a decomposition of the tangent space at every point into the flow direction, an expanding direction, and a contracting direction.

Remark 3.0.2

The geodesic flow on the hyperbolic plane (and any compact, connected factor of the hyperbolic plane) is Anosov. To see this, observe what happens upon wiggling the forward/backward endpoint of the geodesic an element of the unit tangent bundle picks out. The same argument holds in higher dimensions.

Corresponding to Anosov flows are foliations into stable and unstable submanifolds. Define the stable/unstable sets by

$$W^s(x) = \{y \in M \mid d(\phi^t x, \phi^t y) \to 0 \text{ as } t \to \infty\};$$
$$W^u(x) = \{y \in M \mid d(\phi^{-t} x, \phi^{-t} y) \to 0 \text{ as } t \to \infty\}.$$

Then the distribution E^s is the tangent space to the submanifold W^s, and likewise E^u is for the submanifold W^u.

4 Invariant Measures and Ergodic Theory

We now bring in the measure-theoretic preliminaries to discuss the statistically chaotic nature of geodesic flows in negative curvature.

Definition 4.0.1: Measure-Preserving and Ergodic Flows

A measurable flow ϕ^t on a measure space $(X, \mathcal{B}, \mu)$ is said to be *measure-preserving* if $\mu(\phi^t A) = \mu(A)$ for all $t \in \mathbb{R}$, $A \in \mathcal{B}$.

A measure-preserving flow ϕ^t is called *ergodic* (with respect to the measure μ) if the only invariant sets are either measure 0 or comeasure 0. More formally, if we have $\phi^t(A) \subseteq A$ (up to measure 0) for all $t \in \mathbb{R}$, then either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

5 Characterizations of Ergodicity and The Birkhoff Ergodic Theorem

As we have seen, one can characterize ergodicity in terms of invariant sets, but it is also possible to give a characterization in terms of invariant functions.

Proposition 5.0.1

A flow ϕ^t on a measure space $(X, \mathcal{B}, \mu)$ is ergodic if and only if the only essentially ϕ^t-invariant measurable functions are constant mod 0 (in other words, if $f : X \to \mathbb{R}$ is measurable and satisfies $\mu(\{x \in X \mid f(\phi^t x) \neq f(x)\}) = 0$ for every $t \in \mathbb{R}$, then f must be constant almost everywhere).

The following is also an important result concerning ergodic flows.

Theorem 5.0.2: The Birkhoff Ergodic Theorem for Flows

Suppose ϕ^t is a flow on the measure space $(X, \mathcal{B}, \mu)$ and $f \in L^1(X, \mathcal{B}, \mu)$. Then the limits

$$f^+(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\phi^t x) dt \text{ and } f^-(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\phi^{-t} x) dt$$

both exist (in the L^1 sense), are equal μ-almost everywhere, and are ϕ^t-invariant.

Therefore, the flow ϕ^t is ergodic with respect to μ if and only if we have $f^+(x) = f^-(x) = \int_X f d\mu$ for μ-almost every $x \in X$ (for any choice of $f \in L^1$).

The consequences of this theorem are profound. In particular, it implies that ergodicity is equivalent to the *equidistribution* of almost every orbit (a property stronger than density). From a probabilistic point of view, this means that an orbit of a given randomly chosen point has probability $\mu(A)$ of being in a set A in the long-term future.

6 The Liouville Measure

In order to discuss ergodicity, we need to have an underlying measure. (A separate question from the one we will address today is whether, given a flow on a topological space, does there exist a Borel measure on the underlying space with respect to which the flow is ergodic.) There is a natural measure to put on the unit tangent bundle of a Riemannian manifold, which we describe below.

Definition 6.0.1: The Liouville Measure

Suppose M is a Riemannian manifold, so that M comes equipped with a Riemannian volume form (a measure), which we will denote by μ. For each $x \in M$, let λ_x denote the Lebesgue measure on the fiber SM_x. Then define the *Liouville measure* ν on SM by $d\nu(x, v) = dm(x) \times d\mu_x(v)$.

It is not difficult to see that this measure is preserved by the geodesic flow.

7 The Main Theorem

The formal statement is the following.

Theorem 7.0.1: Ergodicity of Geodesic Flow in Negative Curvature

Let M be a compact Riemannian manifold with a smooth metric of negative sectional curvature. Then the geodesic flow $g^t : SM \to SM$ is ergodic with respect to the Liouville measure ν on SM.

The ideas underlying the proof involve considering how a flow-invariant function must behave along the stable and unstable foliations. By combining control of behavior and local-constantness in each direction (as well as along the flow direction), one can deduce that the function must actually be constant almost everywhere, with respect to the Liouville measure. This implies the flow is ergodic.

8 Further Considerations

It turns out that much stronger results may be proven about the dynamics of geodesic flows in this context. For instance, all such flows in this context are actually mixing (of all orders). I can write a sketchy definition on the board.

Furthermore, these systems admit Markov codings, meaning they are in some sense equivalent to symbolic systems. This enables one to study the dynamics of geodesic flows by looking only at symbolic systems.