Jr Geometry: Hyperbolic Groups and the Gromov Geodesic Flow

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September 2024

1 Hyperbolic metric spaces and Quasi-isometries

We begin with a natural generalization of negative curvature to the setting of general metric spaces.

Definition 1.0.1: δ/Gromov hyperbolicity

A proper, geodesic metric space (X, d) is said to be δ*-hyperbolic* if geodesic triangles in X are δ-thin. Formally, there is some $\delta \geq 0$ so that if $\triangle abc \subseteq X$ is a geodesic triangle (all sides of the triangle are geodesics), then $\overline{ab} \subseteq N_\delta(\overline{ac} \cup \overline{bc})$.
If (X, d) is δ-hyperbolic for some $\delta \geq 0$, we call X *Gromov hyperbolic.*

Example Some examples of δ-hyperbolicity:

- Trees are 0-hyperbolic
- $\mathbb{H}^n$ is $\operatorname{arctanh}(1/\sqrt{2})$-hyperbolic
- Riemannian manifolds of negative sectional curvature are always Gromov hyperbolic for some δ (depending on how negatively curved)
- Spaces of bounded diameter (e.g. finite or compact spaces) are always Gromov hyperbolic
- Euclidean spaces are not Gromov hyperbolic

The natural notion of equivalence for such spaces is that of quasi-isometry.

Definition 1.0.2: Quasi-isometry

A map $f : X \to Y$ is a *quasi-isometric embedding* if there exist $K \geq 1$, $C \geq 0$ for which

$$\frac{1}{K} d_X(a, b) - C \leq d_Y(f(a), f(b)) \leq K d_X(a, b) + C.$$

A (K, C) quasi-isometric embedding $f : X \to Y$ is simply said to be a *quasi-isometry* if for all $y \in Y$, there is some $x \in X$ for which $d(f(x), y) \leq C$.

The following are the main important properties of quasi-isometries.

Lemma 1.0.3: Milnor-Svarc Lemma

Suppose $\Gamma \curvearrowright X$ properly discontinuously, cocompactly and by isometries. Then Γ is finitely generated, and the orbit map $\Gamma \to X$ by $\gamma \mapsto \gamma(x_0)$, for all $\gamma \in \Gamma$ and some fixed $x_0 \in X$, is a quasi-isometry.

Lemma 1.0.4: Morse Lemma/Fellow Traveller Property

Fix K, C, δ. There exists R such that if X is δ-hyperbolic, $f : [a, b] \to X$ is a (K, C) quasi-isometric embedding, and L is a geodesic joining $f(a)$ and $f(b)$, then the Hausdorff distance between L and $f([a, b])$ is at most R.

For Gromov hyperbolic spaces, the following corollary is vital.

Corollary 1.0.5

Suppose Y is Gromov hyperbolic and $f : X \to Y$ is a quasi-isometric embedding. Then X is Gromov hyperbolic. Since every quasi-isometry has a quasi-inverse that is also a quasi-isometry, Gromov hyperbolicity is an invariant of quasi isometries of proper geodesic spaces.

2 The Boundary

Associated to a Gromov hyperbolic space X is a natural notion of the boundary. The following is one way to construct it, in the case of working with proper metric spaces.

Definition 2.0.1: The Gromov Boundary

Let X be δ-hyperbolic, and define $G(X) = \{\alpha : [0, \infty) \to X \mid \alpha \text{ an isometric embedding}\}$. Say $\alpha \sim \beta$ if $\sup\{d(\alpha(t), \beta(t))\} < \infty$ (α, β are said to be *asymptotic* in this case; $\sim$ is an equivalence relation). Then define $\partial X = \partial_\infty X = G(X)/\sim$.

Remark 2.0.2

- ∂X can also be formalized by only considering rays emanating from a fixed $x_0 \in X$.
- There is a natural topology on ∂X just by considering convergence of sequences of geodesic rays starting at a given point (by taking subsequences if needed).
- Further, we can topologize $X \cup \partial X$ by saying $x_n \to [\alpha]$ if there is $x_0 \in X$ with a sequence of directed geodesics from x_0 to x_n which (up to taking subsequences) converges to a geodesic ray in $[\alpha]$.

Proposition 2.0.3

- If X is δ-hyperbolic, then ∂X is compact and $X \cup \partial X$ is compact.
- If X, Y are δ-hyperbolic and $f : X \to Y$ is a quasi-isometric embedding, there is a continuous injective map $\partial f : \partial X \to \partial Y$ for which if $x_n \to z$, then $f(x_n) \to \partial f(z)$. Further, if in this situation f is a quasi-isometry, then ∂f is a homeomorphism.

Example

- The boundary of the regular n-valent tree is the Cantor set
- $\partial \mathbb{H}^n \cong S^{n-1}$

3 Hyperbolic groups

Definition for finitely generated groups, independent of choice of generating set.

Definition 3.0.1: Hyperbolic Groups

A finitely generated group G is said to be *hyperbolic* if there is a finite generating set $S \subseteq G$ for which $\mathrm{Cay}(G, S)$ (equipped with the word metric) is Gromov hyperbolic.

Theorem 3.0.2

Suppose S, S' are both finite generating sets for the group G. Then $\mathrm{Cay}(G, S)$ is Gromov hyperbolic if and only if $\mathrm{Cay}(G, S')$ is Gromov hyperbolic.

Example

- F_2 is hyperbolic, and ∂F_2 is the Cantor set
- $\mathbb{Z}$ is hyperbolic, $\partial \mathbb{Z} = \{-\infty, \infty\}$
- $\mathbb{Z}^d$ is not hyperbolic for $d > 1$
- $\pi_1(M)$ where M is a negatively curved Riemannian manifold is hyperbolic
- Suppose $\Gamma \curvearrowright \mathbb{H}^n$ properly discontinuously, cocompactly, and by isometries. Then $M \cong \mathbb{H}^n/\Gamma$ is a hyperbolic manifold, Γ is hyperbolic, $\Gamma \cong \pi_1(M)$, and $\partial \Gamma = S^{n-1}$
- Subgroups of hyperbolic groups are hyperbolic

4 The Geodesic Flow

In the classic setting, given a Riemannian manifold M, one can define the geodesic flow on the unit tangle bundle T^1M by flowing along the unique geodesic through a point with a given tangent vector. Gromov showed that in many cases, the geodesic flow can be modeled purely by using group-theoretic information about $\pi_1(M)$.

Suppose Γ is a hyperbolic group. Start by considering the space

$$\mathcal{G}(\Gamma) = \{c : \mathbb{R} \to C_\Gamma \text{ an isometric embedding}\}.$$

Note that for $\mathbb{H}^n$, the space of isometric embeddings of $\mathbb{R}$ is already $T^1\mathbb{H}^n$, so this is a natural generalization to consider. Furthermore, if $\Gamma \curvearrowright \mathbb{H}^n$ properly discontinuously, cocompactly by isometries, then $\Gamma \curvearrowright T^1\mathbb{H}^n$ in a natural way, and if $M \cong \mathbb{H}^n/\Gamma$, then $T^1M \cong T^1\mathbb{H}^n/\Gamma$.

Note that $\mathcal{G}(\Gamma)$ admits actions by $\Gamma, \mathbb{Z}_2, \mathbb{R}$: $(\gamma, c) \mapsto \gamma \cdot c$, time-reversal, and $(s, c(t)) \mapsto c(t+s)$, respectively.

For $T^1\mathbb{H}^n$, we have an identification $T^1\mathbb{H}^n \cong (\partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \triangle) \times \mathbb{R}$. We would like to perform an analogous identification here, but the issue is that **there may not be a unique geodesic joining two points in the boundary** for general hyperbolic groups. Thus, we need to "collapse" $\mathcal{G}(\Gamma)$ via an equivalence relation identifying geodesics with the same endpoints.

Theorem 4.0.1: (Gromov, Champetier, Mineyev)

Let Γ be a hyperbolic group, $U(\Gamma) = (\partial\Gamma \times \partial\Gamma \setminus \Delta) \times \mathbb{R}$. There exists a complete metric on $U(\Gamma)$, a flow $\varphi_t : U(\Gamma) \to U(\Gamma)$, and a properly discontinuous, cocompact action of $\Gamma \curvearrowright U(\Gamma)$ by isometries for which

- The map $t \mapsto (z, w, t) \in U(\Gamma)$ is a quasi-isometric embedding.
- $\varphi_t(z, w, s) = (z, w, s+t)$, and each φ_t is bi-Lipschitz
- The action of Γ commutes with φ_t, and the flow descends to a topologically transitive flow $\hat{\varphi}_t$ on $\hat{U}(\Gamma)$.
- For $\gamma \in \Gamma$, there is a function $c_\gamma : \partial\Gamma \times \partial\Gamma \setminus \Delta \to \mathbb{R}$ such that $\gamma(x, y, t) = (\gamma x, \gamma y, c_\gamma(x, y) + t)$.

Corollary 4.0.2: Generalization of the classical geodesic flow

Suppose M is a closed, negatively curved Riemannian manifold and $\Gamma = \pi_1(M)$. Then the flow $\mathbb{R} \curvearrowright \hat{U}(\Gamma)$ is topologically conjugate to the geodesic flow on T^1M (in other words, there is an $\mathbb{R}$-equivariant homeomorphism between these spaces).

The study of dynamical properties of the classical geodesic flow has a whole history of its own. It is natural to ask for an analogous understanding of the geodesic flow associated to a hyperbolic group.

5 Anosov representations

Open questions surrounding the geodesic flow of general hyperbolic groups, what is established for groups admitting Anosov reps.

Classical geodesic flows for negatively curved manifolds are known to be Anosov, meaning the tangle bundle admits a splitting into expanding/contracting directions. This is a fundamental feature of hyperbolic dynamics. In general, it is not known whether the geodesic flow of a hyperbolic group satisfies the same properties. However, those admitting an "Anosov representation" do.

Definition 5.0.1: Anosov representations

If $k \in [1, d/2] \cap \mathbb{Z}$, we say that a representation $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ (with Γ finitely generated) is P_k-*Anosov* if there exists K, C so that

$$\frac{1}{K} d(id, g) - C \leq \log \frac{\sigma_k(\rho(g))}{\sigma_{k+1}(\rho(g))} \leq K d(id, g) + C$$

for all $g \in \Gamma$.

If a finitely generated group Γ admits an Anosov representation, it turns out that Γ is necessarily hyperbolic.

Theorem 5.0.2

If Γ is a hyperbolic group admitting an Anosov representation, then its geodesic flow is metric Anosov.

It is not known whether the geodesic flow of general hyperbolic group is metric Anosov.