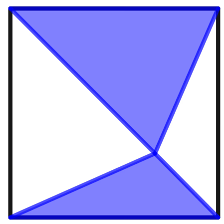
UT SMMG Geometry Puzzles

October 2024

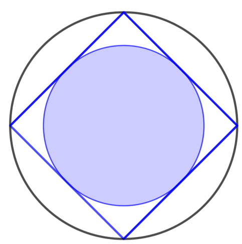
Geometry in the Plane

These puzzles come courtesy of John Carlos Baez and Alex Bellos.

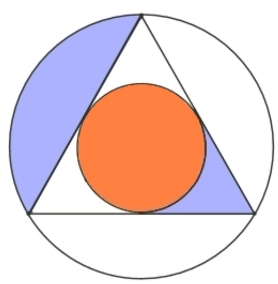
1. A point inside a square is connected to its four vertices. What fraction of the square is shaded?



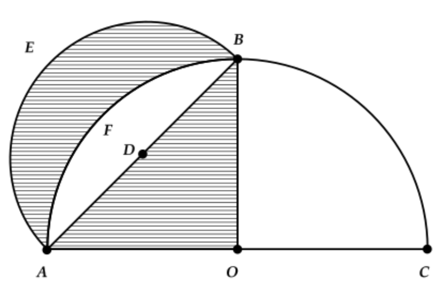
2. A square is shown with its inscribed and circumscribed circle. What fraction of the larger circle is shaded?



3. Show that the orange and purple regions have the same area.



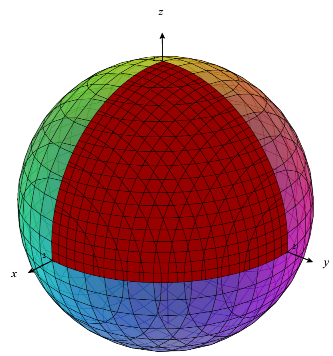
4. Show that the two shaded regions have the same area. (This one is hard!)



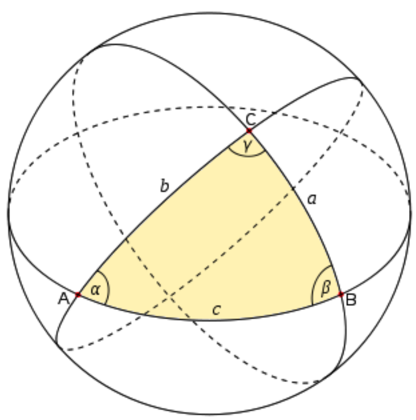
Geometry on the Globe

1. How would you describe shortest paths between points on a sphere?

2. Consider the triangle that takes up $\frac{1}{8}$ of the sphere (shown below). What is the sum of the angles in this triangle? What is the area of this triangle?



3. Girard's Theorem: Can you come up with a formula for the surface area of a spherical triangle like the one shown below?



(Hint: the answer should only depend on the sum of angles in the triangle and the radius of the sphere)

4. Based on your answers for the previous exercises, can you guess a basic fact about what must always be true for sums of angles in spherical triangles?

Constructible Numbers

Start with two points in the plane, called O, A (a distance 1 apart). You are given two tools to construct new points: a **straightedge** and **compass**.

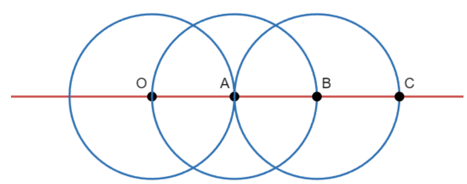
Rules

1. **Straightedge**: can be used to draw line segments between two points or to extend an existing line segment. It has no markings.
2. **Compass**: can form circles with a given center and point on the circle. They can be arbitrarily large.

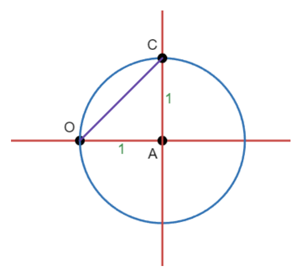
A point B in the plane is said to be **constructible** if B can be obtained from a straightedge and compass construction in finitely many steps, starting with just the points O, A. A number r is said to be constructible if there is a segment of length r that is constructible in finitely many steps.

Examples

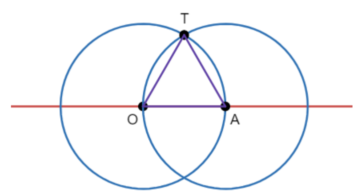
Every whole number is constructible:
The length from $\overline{OB}$ is twice that of $\overline{OA}$, so we say 2 is constructible (since 1 is the length of $\overline{OA}$). Likewise, the length of $\overline{OC}$ is 3 times that of $\overline{OA}$, hence 3 is constructible, and so on.



$\sqrt{2}$ is constructible:
The length of the purple line, $\overline{OC}$ is $\sqrt{2}$ (by the Pythagorean theorem). This means $\sqrt{2}$ is constructible!



Equilateral triangles are constructible:
The triangle $\triangle OAT$ is equilateral since all of its sides have the same length.



Problems

1. Construct a regular hexagon. (Hint: use the equilateral triangles construction in the process.)
2. Construct $\sqrt{3}$. (Hint: use a special triangle.)
3. Construct an angle bisector (given two lines meeting at angle θ, construct lines meeting at angle $\theta/2$).
4. Do you think there are regular polygons that are not constructible?
5. Try to determine whether or not π is constructible.