Frobenius' Theorem on Integrable Distributions

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Guiding Question

Suppose M is a smooth manifold.

If we are given a k-dimensional subbundle of TM (called a **distribution** on M), is there a k-dimensional subbundle of M (called an **integral manifold**) whose tangent space at each point is the given subspace?

Some Formalism

We say a distribution is of **rank** k if it a rank k subbundle of TM, and similarly it is said to be **smooth** if it is a smooth subbundle.

Suppose $D \subseteq TM$ is a smooth distribution. If $N \subseteq M$ is a nonempty immersed submanifold, it is called an **integral manifold of** D provided that $T_pN = D_p$ for every $p \in N$.

Conditions of Involutivity and Integrability

A smooth distribution D on M is called **involutive** if given any two smooth local sections of D, their Lie bracket is also a local section of D.

In other words, given smooth vector fields X, Y defined on an open section of M for which $X_p, Y_p \in D_p$ for all p, we have that $[X, Y]_p \in D_p$ for every p.

We also say that a smooth distribution D on M is **integrable** if each point of M is contained in an integral manifold of D.

Necessary Condition

Lemma

Every integrable distribution is involutive.

The proof is straightforward.

Let $D \subseteq TM$ be a smooth distribution, and suppose X, Y are smooth local sections of D defined on some open $U \subseteq M$. Let $p \in U$, and N be an integral manifold of D containing p. The fact that X and Y are sections of D means that X and Y are tangent to N. But then it is also true that $[X, Y]$ is tangent to N, and therefore $[X, Y]_p \in D_p$, thus D is involutive.

Examples of Involutivity and Non-Involutivity

Two involutive examples:

- If V is a nowhere-vanishing smooth vector field on M, then V spans a smooth rank-1 distribution on M. The image of any integral curve of V is an integral manifold of D.
- In $\mathbb{R}^n$, the vector fields $\partial/\partial x^1, \ldots, \partial/\partial x^k$ span a smooth distribution of rank k. The k-dimensional affine subspaces parallel to $\mathbb{R}^k$ are the integral submanifolds.

Non-involutive example:

- Let D be the smooth distribution on $\mathbb{R}^3$ spanned by the vector fields

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \text{and} \quad Y = \frac{\partial}{\partial y}$$

and it turns out D has no integral manifolds.

Criteria for Involutivity

Lemma (Lie Subalgebra Criterion)

Let $\mathfrak{X}(M)$ denote the space of smooth vector fields on M. Let $D \subseteq TM$ be a smooth distribution, and let $\Gamma(D) \subseteq \mathfrak{X}(M)$ be the space of smooth global sections of D. Then D is involutive $\iff$ $\Gamma(D)$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Lemma (Local Frame Criterion)

Let $D \subseteq TM$ be a distribution. If in a neighborhood of every point of M there exists a smooth local frame $(V_1, \ldots, V_k)$ for D such that $[V_i, V_j]$ is a section of D for all pairs i, j, then D is involutive.

Criteria for Involutitvity (Continued)

We say a p-form $\omega \in \Omega^p(M)$ *annihilates* D if $\omega(X_1, \ldots, X_p) = 0$ whenever $X_1, \ldots, X_p$ are local sections of D. Let $\ell^p(D)$ be the space of smooth p-forms which annihilate D.

Lemma (1-Form Criterion)

Let $D \subseteq TM$ be a smooth distribution. Then D is involutive if and only if the following condition is satisfied: given any smooth 1-form η that annihilates D on an open $U \subseteq M$, then $d\eta$ also annihilates D on U.

Let $\Omega^*(M)$ denote the graded algebra of smooth differential forms on a smooth n-manifold M. An ideal $\ell \subseteq \Omega^*(M)$ is called a differential ideal if $d(\ell) \subseteq \ell$.

Lemma (Differential Ideal Criterion)

Let M be a smooth manifold. A smooth distribution $D \subseteq TM$ is involutive if and only if $\ell(D) = \bigoplus_{p=0}^{n} \ell^p(D)$ is a differential ideal in $\Omega^*(M)$.

Complete Integrability

A smooth chart (U, ϕ) on M is flat for D if $\phi(U)$ is a cube in $\mathbb{R}^n$, and at points of U, D is spanned by the first k coordinate vector fields $\partial/\partial x^1, \ldots, \partial/^k$. In any such chart, each slice of the form $x^{k+1} = c^{k+1}, \ldots, x^n = c^n$ for constants $c^{k+1}, \ldots, c^n$ is an integral manifold of D.

We say a distribution $D \subseteq TM$ is **completely integrable** if there exists a flat chart for D in a neighborhood of each point of M. Every completely integrable distribution is clearly integrable and therefore involutive.

$$\text{Completely Integrable} \implies \text{Integrable} \implies \text{Involutive}$$

Frobenius' Theorem

Frobenius' theorem states that the previous implication is an equivalence.

Theorem (Frobenius)

Every involutive distribution is completely integrable.

This is a statement of central importance in smooth manifold theory.

Foliations

Let M be a smooth n-manifold.

We define $\mathcal{F}$, a **foliation of dimension k on M**, to be a collection of disjoint, connected, nonempty, immersed k-dimensional submanifolds of M (called the **leaves of the foliation**), whose union is M, and such that in a neighborhood of each point $p \in M$ there exists a flat chart for $\mathcal{F}$.

A flat chart (U, ϕ) for $\mathcal{F}$ is one for which $\phi(U)$ is a cube in $\mathbb{R}^n$ and each submanifold in $\mathcal{F}$ intersects U in either the empty set or a countable union of k-dimensional slices of the form $x^{k+1} = c^{k+1}, \ldots, x^n = c^n$.

Global Frobenius Theorem

Theorem (Global Frobenius Theorem)

Let D be an involutive distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of D forms a foliation of M.

Theorem (Lie Subgroups are Weakly Embedded)

Every Lie subgroup is an integral manifold of an involutive distribution, and therefore is a weakly embedded submanifold.

Theorem (The Lie Subgroup Associated with a Lie Subalgebra)

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra, then there is a unique connected Lie subgroup of G whose Lie algebra is $\mathfrak{h}$.

Applications to PDEs

Theorem

Suppose α and β are smooth real-valued functions defined on some open $W \subseteq \mathbb{R}^3$ satisfying that

$$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z}$$

on W. For each $(x_0, y_0, z_0) \in W$, there is a neighborhood U of (x_0, y_0) in $\mathbb{R}^2$ and a unique smooth function $u : U \to \mathbb{R}$ satisfying

$$\frac{\partial u}{\partial x}(x, y) = \alpha(x, y, u(x, y))$$
$$\frac{\partial u}{\partial y}(x, y) = \beta(x, y, u(x, y))$$

References

John M. Lee (2018)
Introduction to Smooth Manifolds
Chapter 19

The End