Clifford Algebras and Spinor Groups

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The Main Result

For $n > 2$, the Lie group $\mathrm{SO}(n)$ has a connected double cover. We will explicitly construct the (compact, connected) Lie group $\mathrm{Spin}(n)$ and a surjective homomorphism from this group to $\mathrm{SO}(n)$ with kernel $\mathbb{Z}_2$. In other words, we will have the following short exact sequence of Lie groups

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 0$$

Strategy of Construction

Our construction follows section 1.6 in Brocker and tom Dieck's text.

To construct the Spinor groups, we will use the theory of real Clifford algebras corresponding to the quadratic form

$$Q : \mathbb{R}^n \to \mathbb{R} \text{ by } Q(x) = -|x|^2$$

Real Clifford Algebras

Definition

Let V be a finite dimensional vector space and $Q : V \to \mathbb{R}$ a quadratic form. The **Clifford Algebra** $C(Q)$ is an $\mathbb{R}$-algebra with unit 1 with

- A linear **structure map** $i : V \to C(Q)$ for which $(i(x))^2 = Q(x) \cdot 1$
- A **universal property** (next slide)

Universal Property

Let A be any $\mathbb{R}$-algebra with 1 and a linear map $j : V \to A$ for which $(j(x))^2 = Q(x) \cdot 1$. Then there is a unique **universal homomorphism** $\kappa_j : C(Q) \to A$ making the following diagram commutative:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "j"'] & C(Q) \arrow[d, dashed, "\kappa_j"] \\ & A \end{tikzcd}$$

For a given (V, Q) there is (up to isomorphism) only one pair $(C(Q), i)$ which will satisfy the above universal property.

Construction of Clifford Algebras

We start with the tensor algebra T of V:

$$T = \bigoplus_{\upsilon=0}^{\infty} V^{(\upsilon)} \text{ where } V^{(0)} = \mathbb{R},\ V^{(1)} = V,\ \text{and } V^{(\upsilon)} = \bigotimes_{k=1}^{\upsilon} V$$

with multiplication induced by setting $a \cdot b = a \otimes b$ (so if $a \in V^{(\upsilon)}$, $b \in V^{(\mu)}$, then $a \cdot b \in V^{(\upsilon+\mu)}$). Let $\mathfrak{a}$ be the ideal generated by $\{x \otimes x - Q(x) \cdot 1 \mid x \in V\}$.

Then $C(Q) = T/\mathfrak{a}$, and the structure map i is the composition

$$i : V \hookrightarrow T \to T/\mathfrak{a}$$

The Particular Case of Concern

If $V = \mathbb{R}^n$ comes equipped with the quadratic form $Q(x) = -|x|^2$, we denote its Clifford algebra $C(Q)$ by C_n. The relations in $\mathfrak{a}$ say that

$$e_\nu^2 = -1 \text{ and } e_\nu \cdot e_\mu = -e_\mu \cdot e_\nu \text{ if } \nu \neq \mu$$

The above is easily derived from $(c_1 e_1 + \ldots + c_n e_n)^2 = -(c_1^2 + \ldots + c_n^2)$.

Further, there are canonical isomorphisms

$$C_0 \cong \mathbb{R}\,,\ C_1 \cong \mathbb{C}\,,\text{ and } C_2 \cong \mathbb{H}$$

Canonical Anti-Automorphism

$C(Q)$ has a canonical anti-automorphism $t : C(Q) \to C(Q)$, $x \mapsto x^t$. It satisfies $(x \cdot y)^t = y^t \cdot x^t$, $t^2 = \mathrm{Id}$, and is uniquely determined by $x^t = x$ for $x \in i(V)$.

Define the opposite algebra by $C(Q)^{\mathrm{op}} = C(Q)$ as real vector spaces, but setting $x \cdot y$ in $C(Q)^{\mathrm{op}}$ equal to $y \cdot x$ in $C(Q)$. In $C(Q)^{\mathrm{op}}$, we still have $x^2 = Q(x) \cdot 1$ for $x \in i(V)$. Thus by the universal property, there is a map

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "i"'] & C(Q) \arrow[d, dashed, "t"] \\ & C(Q)^{\mathrm{op}} \end{tikzcd}$$

and it is an isomorphism, since $C(Q) \to C(Q)^{\mathrm{op}} \to C(Q)^{\mathrm{op\,op}} = C(Q)$ is Id.

Canonical Anti-Automorphism (Continued)

Each element in $C(Q)$ may be expressed (nonuniquely) as a linear combination of elements of the form $x = x_1 \ldots x_k$ with $x_\nu \in i(V)$. The previous diagram shows that $(x_1 \ldots x_k)^t = x_k \ldots x_1$, which completely determines t.

Analogously, there is a canonical automorphism

$$\alpha : C(Q) \to C(Q), \quad \text{with } \alpha^2 = \mathrm{Id}, \quad \text{and } \alpha(x) = -x \text{ for } x \in i(V)$$

Thus we have $\alpha(x_1 \ldots x_k) = (-1)^k (x_1 \ldots x_k)$, when $x_\nu \in i(V)$ again.

$\mathbb{Z}_2$-Grading

For $\nu = 0, 1$, let $C(Q)^\nu$ be the eigenspace for the eigenvalue $(-1)^\nu$ of α. Then

$$C(Q) = C(Q)^0 \oplus C(Q)^1$$

as vector spaces. This is called a $\mathbb{Z}_2$**-grading**, meaning that if we reduce exponents reduced mod 2, we have that

$$x \in C(Q)^\nu,\ y \in C(Q)^\mu \longrightarrow x \cdot y \in C(Q)^{\nu+\mu}$$

$\mathbb{Z}_2$-Grading (Continued)

Let $A = A^0 \oplus A^1$, $B = B^0 \oplus B^1$ be $\mathbb{Z}_2$-graded algebras. Then there is a graded tensor product, $A \hat{\otimes} B$, defined by

$$(A \hat{\otimes} B)^0 = (A^0 \otimes B^0) \oplus (A^1 \otimes B^1) \text{ and } (A \hat{\otimes} B)^1 = (A^0 \otimes B^1) \oplus (A^1 \otimes B^0)$$

with multiplication given by $(a' \otimes b) \cdot (a \otimes b') = (-1)^{\nu\mu}(a' \cdot a) \otimes (b \otimes b')$, for $a \in A^\mu$ and $b \in B^\nu$.

Unsurprisingly, $A \hat{\otimes} B$ is again a $\mathbb{Z}_2$-graded algebra.

Important Isomorphism

Lemma

Let V, W be finite-dimensional vector spaces with corresponding quadratic forms P, Q and structure maps i, j. Then $V \oplus W$ has the quadratic form $(P \oplus Q)(v, w) = P(v) + Q(w)$. Define the linear map

$$f : V \oplus W \to C(P) \hat{\otimes} C(Q) \;\; by \;\; (v, w) \mapsto i(v) \otimes 1 + 1 \otimes j(w)$$

Then f induces an isomorphism (also denoted by f)

$$f : C(P \oplus Q) \to C(P) \hat{\otimes} C(Q)$$

The proof is straightforward, just making use of the universal property again and explicitly constructing the inverse homomorphism. Note that this lemma immediately implies $C_n = \mathbb{C} \hat{\otimes} \ldots \hat{\otimes} \mathbb{C}$ (n factors).

Important Conclusion

Corollary

The structure map is injective. Let V have basis $e_1, \ldots, e_n$. Then the products

$$e_{\upsilon_1} \ldots e_{\upsilon_k} \text{ where } 1 \leq \upsilon_1 < \ldots < \upsilon_k \leq n$$

and 1 form a basis of the real vector space $C(Q)$. In particular, the aforementioned relations ($e_\upsilon^2 = -1$, $e_\upsilon \cdot e_\mu = -e_\mu \cdot e_\upsilon$) form a complete set of relations for these products, and $C(Q)$ has dimension 2^n.

Proof of Corollary

We proceed by induction on the dimension of V. Note first that if V is n-dimensional, we can diagonalize any quadratic form over $\mathbb{R}$, i.e. we have the orthogonal splitting $(V, Q) = \bigoplus_{k=1}^{n}(V_k, Q_k)$ where each V_k is one dimensional.

If $\dim V = 1$, the tensor algebra is $T = \mathbb{R}[X]$ where $i(e_1) = X$, so that $C(Q) = \mathbb{R}[X]/\langle X^2 - Q(e_1)\rangle$, and in this case the statement is clear.

In general, the relation $Q(e_\nu + e_\mu) - Q(e_\nu) - Q(e_\mu) = e_\nu e_\mu + e_\mu e_\nu$ shows that the stated elements generate $C(Q)$.

Conjugation on $C(Q)$ is given by $t\alpha = \alpha t : C(Q) \to C(Q)$, denoted $x \mapsto \overline{x}$ (note $t\alpha = \alpha t$ on V so it holds on $C(Q)$). This is an algebra anti-automorphism. Let $C(Q)^* \subset C(Q)$ be the group of units and define

$$\Gamma(Q) = \{x \in C(Q)^* \mid \alpha(x) \cdot v \cdot x^{-1} \in V, \text{ for all } v \in V\}$$

$\Gamma(Q)$ is a subgroup of $C(Q)^*$ and is called the **Clifford group** of Q (it's a group because α is an automorphism and V is finite dimensional). Also, α and t incude an automorphism and anti-automorphism of $\Gamma(Q)$.

The map $N : C(Q) \to C(Q)$ by $x \mapsto x \cdot \overline{x}$ is called the **norm**, and for $x \in V$ we have $N(x) = -Q(x) \cdot 1$.

Homomorphism of Γ_n

For the Clifford algebras C_n, let Γ_n denote the Clifford group. This comes with a homomorphism

$$\rho : \Gamma_n \to \mathrm{Aut}(\mathbb{R}^n) \text{ by } \rho(x)v = \alpha(x)vx^{-1} \text{ for } x \in \Gamma_n,\ v \in \mathbb{R}^n$$

Lemma (1)

$\mathbb{R}^n \setminus \{0\} \subset \Gamma_n$, *and if* $x \in \mathbb{R}^n \setminus \{0\}$, *then* $\rho(x)$ *is the reflection in the hyperplane orthogonal to* x.

Taking $x = e_1$, we can just compute that $\rho(e_1)e_1 = -e_1$ and $\rho(e_1)e_k = e_k$.

Lemma (2)

The kernel of ρ *is the group of nonzero real multiples of* $1 \in C_n$, *i.e.* $\mathbb{R}^*$.

Proof of Lemma 2

Superscripts of k denote elements of C_n^k.

Let $x \in \ker(\rho)$, and we have $\alpha(x) \cdot v = v \cdot x$ for every $v \in \mathbb{R}^n$.

We can express $x = x^0 + x^1$, so $x^0 v = v x^0$, $-x^1 v = v x^1$.

For any υ, write $x^0 = a^0 + e_\upsilon b^1$ where each of a^0, b^1 doesn't contain a factor of e_υ. We have $x^0 = e_\upsilon^{-1} x^0 e_\upsilon$, and therefore because $a^0 e_\upsilon = e_\upsilon a^0$ and $e_\upsilon b^1 = -b^1 e_\upsilon$, we have $x^0 = a^0 - e_\upsilon b^1$. Thus $e_\upsilon b^1 = 0$, meaning x^0 has no monomial with a factor of e_υ, i.e. $x^0 \in \mathbb{R} \cdot 1$.

Similarly, if $x^1 = a^1 + e_\upsilon b^0$, the above relations give $x^1 = a^1 - e_\upsilon b^0$, so that $x^1 \in \mathbb{R}$, and thus $x^1 = 0$ because $\mathbb{R} \subset C_n^0$.

Therefore, $x = x^0 \in \mathbb{R} \cap \Gamma_n = \mathbb{R}^*$.

Properties of the Norm

Lemma

If $x \in \Gamma_n$*, then* $N(x) \in \mathbb{R}^*$.

Follows by showing $\rho(N(x)) = \mathrm{Id}$ since t is the identity on $\mathbb{R}^n$.

Lemma

$N|_{\Gamma_n} : \Gamma_n \to \mathbb{R}^*$ *is a homomorphism and* $N(\alpha(x)) = N(x)$.

Follows by directly computing that $N(xy) = N(x)N(y)$ and that α commutes with N.

Corollary

$\rho(\Gamma_n) \subset O(n)$.

$N(\rho(x)v) = N(\alpha(x)vx^{-1}) = N(\alpha(x)) \cdot N(v) \cdot N(x^{-1}) = N(v)$ so $\rho(x) \in \mathrm{O}(n)$.

Last Properties

Define $\mathrm{Pin}(n)$ to be the kernel of $N : \Gamma_n \to \mathbb{R}^*$ for $n \geq 1$.

Theorem

The map $\rho|_{Pin(n)}$ has image $O(n)$ and kernel generated by $-1 \in C_n$. Thus we have an exact sequence of Lie groups

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Pin(n) \longrightarrow O(n) \longrightarrow 0$$

This follows since any orthogonal transformation can be expressed as a product of reflections, and all reflections are in the image of $\rho|_{\mathrm{Pin}(n)}$. Also, $\ker(\rho|_{\mathrm{Pin}(n)}) = \ker(\rho) \cap \ker(N) = \{t \in \mathbb{R}^* \mid N(t) = 1\} = \{1, -1\}$.

Main Result

Define $\mathrm{Spin}(n) \subset \mathrm{Pin}(n)$ to be the preimage of $\mathrm{SO}(n)$ under $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$.

Therefore, we have an exact sequence of Lie groups

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 0$$

Theorem

For $n \geq 2$, the homomorphism $\rho : Spin(n) \to SO(n)$ is a nontrivial double covering.

Follows since $w : t \mapsto \cos(t) + \sin(t)e_1 \cdot e_2$ for $0 \leq t \leq \pi$ is an arc in $\mathrm{Pin}(n)$ connecting $-1, 1$ which consistute the kernel of ρ.

Last Corollaries

Corollary

The fundamental group $\pi_1(SO(n))$ is isomorphic to $\mathbb{Z}_2$ for $n \geq 3$, so $Spin(n)$ is simply connected and $\rho : Spin(n) \to SO(n)$ is the universal covering.

Finally, there is a standard isomorphism of groups $\kappa : \mathrm{SU}(2) = \mathrm{Sp}(1) \to \mathrm{Spin}(3)$, and the canonical projection $\pi : \mathrm{Sp}(1) \cong \mathrm{Spin}(3) \to \mathrm{SO}(3)$ is the adjoint representation of $\mathrm{Sp}(1)$.

References

Brocker and tom Dieck (1985)
Representations of Compact Lie Groups
Section 1.6

The End