The Pressure Metric for Hitchin Components

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Outline

Teichmüller Space, Hyperbolic Structures, and Fricke Space

Let S be a closed, oriented smooth surface of genus $g \geq 2$. Define

$$\mathcal{T}(S) = \left\{\begin{matrix}\text{Riemann surface}\\ \text{structures on S}\end{matrix}\right\} \qquad \mathrm{Hyp}(S) = \left\{\begin{matrix}\text{Hyperbolic}\\ \text{metrics on S}\end{matrix}\right\}$$

$$\mathcal{F}(S) = \left\{\begin{matrix}\rho\colon \pi_1 S \to \mathbf{PSL}_2(\mathbb{R})\\ \text{discrete, faithful}\end{matrix}\right\}$$

By the Riemann Uniformization Theorem, we have

Theorem

$$\mathcal{T}(S) \cong Hyp(S) \cong \mathcal{F}(S) \cong \mathbb{R}^{6g-6}$$

Many features of interest: Teichmüller metric, Weil-Petersson metric, twist flow, earthquake flow, $\mathbf{SL}_2(\mathbb{R})$ action, compactifications, nice coordinates, etc.

Discrete Subgroups of Lie Groups

Γ - discrete, finitely generated group

G - Lie group

How does $\Gamma \hookrightarrow G$? Deformation spaces?

A **continuous deformation** of Γ in G is a path $(\rho_t)_{t\in[0,1)} \subseteq \mathrm{Hom}(\Gamma, G)$.

Dichotomy: rigidity versus flexibility

- Lattices (finite covolume): usually few deformations
- Surface groups in rank 1: many deformations

Rigidity Phenomena

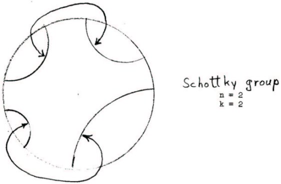
Classical rigidity theorems:

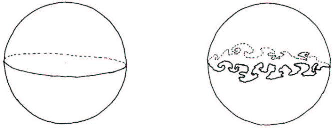
- Local rigidity
  - Lattices in many G only have trivial deformations
- Mostow Rigidity
  - All isomorphic lattices in $\mathrm{Isom}^+(\mathbb{H}^n)$ ($n \geq 3$) are conjugate
- Margulis Superrigidity
  - Representations of discrete subgroups extend to representations of G

Examples of Flexibility

- Schottky groups
  - Representations of free groups $\mathbb{F}_m \to \mathbf{PSL}_2(\mathbb{R})$ (ping-pong on $\partial\mathbb{H}^2$)
- Quasi-Fuchsian groups
  - Deform from Fuchsian representations $\pi_1(S) \hookrightarrow \mathbf{PSL}_2(\mathbb{R}) \subseteq \mathbf{PSL}_2(\mathbb{C})$
- Thurston's bending
  - Deform from lattices $\Gamma \leqslant \mathbf{SO}(n, 1)$ in $\mathbf{SO}(n + 1, 1)$ under $A \mapsto 1 \oplus A$

Pictures from Thurston's notes:





DEFINITION 8.7.1. The Kleinian group Γ is called a *quasi-Fuchsian group* if L_Γ is topologically S^1.

These are all **convex cocompact** representations into rank 1 Lie groups

Higher Fricke Spaces

Let G be a real Lie group of rank ≥ 2.

Definition (Higher Fricke Spaces)

A **higher Fricke space** is a subset $\mathcal{F}_G(S) \subseteq \mathrm{Hom}(\pi_1(S), G)/G$, which is a union of components consisting of discrete and faithful representations.

Immediate questions:

- Do such things exist? Under what conditions on G?
- If they do exist, is there geometric significance?
- Analogies with the structrue of $\mathcal{F}(S)$?

This area has been dubbed **higher Teichmüller-Thurston theory**

For this talk: just $\mathbf{SL}_d(\mathbb{R})$ or $\mathbf{PSL}_d(\mathbb{R})$ (for $d \geq 3$)

The Hitchin Component and Fuchsian Locus

Unique irreducible representation $\tau_d : \mathbf{SL}_2(\mathbb{R}) \to \mathbf{SL}_d(\mathbb{R})$ for any $d \geq 2$.

Definition

The connected components $\mathcal{H}_d(S) \subseteq \mathrm{Hom}(\pi_1(S), \mathbf{PSL}_d(\mathbb{R}))/\mathbf{PGL}_d(\mathbb{R})$ containing representations which factor as

$$\pi_1(S) \xrightarrow{\rho_0} \mathbf{PSL}_2(\mathbb{R}) \xrightarrow{\tau_d} \mathbf{PSL}_d(\mathbb{R})$$

(with $\rho_0 \in \mathcal{F}(S)$) are known as **Hitchin components**.
The set $\tau_d(\mathcal{F}(S)) \subseteq \mathcal{H}_d(S)$ is known as the **Fuchsian locus**.

Longstanding question: do Hitchin components hold geometric significance for S (known for $\mathcal{F}(S)$, as well as $\mathcal{H}_3(S)$ due to Goldman-Choi '93)

Structure of The Hitchin Component

Using techniques of Higgs bundles (no geometry), Hitchin showed

Theorem (Hitchin '92)

$\mathcal{H}_d(S)$ is an analytic manifold and there is a diffeomorphism

$$\mathcal{H}_d(S) \cong \mathbb{R}^{-\chi(S)\dim(\mathbf{PSL}_d(\mathbb{R}))} = \mathbb{R}^{(2g-2)(d^2-1)}.$$

Labourie then uncovered

Theorem (Labourie '06)

If $\rho \in \mathcal{H}_d(S)$, then

- *ρ is discrete and faithful, and irreducible;*
- *Orbit map $\tau_\rho : \pi_1(S) \to \mathbf{PSL}_d(\mathbb{R})/\mathbf{PSO}_d(\mathbb{R})$ is a quasi-isometric embedding*
- *If $\gamma \in \pi_1(S)$, with $\gamma \neq e$, then $\rho(\gamma)$ is diagonalizable (over $\mathbb{R}$) and has distinct eigenvalues.*

Moving to Higher Rank

	Classical	**Mysterious**
Lie groups:	Rank 1	Rank ≥ 2
Symmetric spaces:	All flats: 1 dim'l	Flats of dim ≥ 2
Deformation spaces:	Teichmüller space	Hitchin components
Representations:	Convex cocompact	Anosov
Interpretation:	Hyperbolic structures	Dynamical systems
Metrics:	Weil-Petersson metric	*Pressure metrics*

Convex Cocompactness in Rank 1

Definition (Convex Cocompactness)

A representation $\rho : \Gamma \to \mathbf{PO}(n, 1) = \mathrm{Isom}^+(\mathbb{H}^n)$ is **convex cocompact** if either of the following equivalent conditions hold:

1. The orbit map $\tau_\rho(\gamma) = \rho(\gamma)(x_0)$ is a quasi-isometric embedding $\Gamma \hookrightarrow \mathbb{H}^n$;
2. $\rho(\Gamma)$ preserves a convex domain $\Omega \subseteq \mathbb{H}^n$ and $\Omega/\rho(\Gamma)$ is compact.

In higher rank, this equivalence falls apart, and naive generalizations fail

Quasi-isometry alone: too flexible, lacks stability
Convex cocompactness of the action: too rigid, few examples/deformations

Generalizing via Dynamical Behavior

Convex cocompact representations come with continuous, dynamics-preserving, injective ρ-equivariant maps $\xi_\rho : \partial\Gamma \to \partial\mathbb{H}^n$

A focus on the dynamics at infinity of a representation ρ generalizes nicely

Insist on existence of limit maps like these $\Longrightarrow$ Anosov representations

Anosov Representations

A simple description of an appropriate generalization:

Definition (Bochi-Potrie-Sambarino, Kapovich-Leeb-Porti)

Let $\rho : \Gamma \to \mathbf{SL}_n(\mathbb{R})$ a representation. Let $\sigma_k(\rho(\gamma))$ denote the kth singular value of $\rho(\gamma) \in \mathbf{SL}_n(\mathbb{R})$. Then ρ is P_k-**Anosov** if there exist $K, C > 0$ so that

$$\frac{1}{K} d(e, \gamma) - C \leq \log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \leq K d(e, \gamma) + C$$

for all $\gamma \in \Gamma$, where d is a word metric on Γ.

From this point on, everything will work for P_1-Anosov representations

The Dynamical Point of View

Labourie's pioneering work:

Points in Fricke space and the Hitchin component are **dynamical systems**.

Fix $\rho : \pi_1(S) \to \mathbf{SL}_d(\mathbb{R})$, define $E_\rho = (T^1\mathbb{H}^2 \times \mathbb{R}^d)/\pi_1(S)$

Theorem (Labourie '06)

If $\rho \in \mathcal{H}_d(S)$, then there is a geodesic-flow-invariant splitting

$$E_\rho = L_1 \oplus L_2 \oplus \cdots \oplus L_d$$

into line bundles so that the flow is contracting on $L_i \otimes L_j^*$ if $i > j$.

Anosov-ness in the strongest sense: singular value gaps at every index

Overview of the Plan

1. Bonahon: build Weil-Petersson with geometric intersection
2. McMullen: metric from thermodynamic formalism $\implies$ Weil-Petersson
3. Bridgeman-Canary-Labourie-Sambarino: bridged these two perspectives

Geodesic Currents, Intersection, and Weil-Petersson

Definition (Geodesic Currents)

A **geodesic current** is a $\pi_1(S)$-invariant measure on $\mathcal{G}(\tilde{S}) \cong (S^1 \times S^1 \setminus \Delta) \times \mathbb{R}$. Let $C(S)$ denote the collection of geodesic currents, with the weak-$*$ topology.

- There are embeddings $\mathcal{F}(S) \hookrightarrow C(S)$ and $\mathbb{R}^+[\pi_1(S)] \hookrightarrow C(S)$;
- Symmetric, bilinear intersection form $i : C(S) \times C(S) \to \mathbb{R}^{\geq 0}$ generalizing geometric intersection for geodesics;
- i locally minimized on the diagonal $\mathcal{F}(S) \xhookrightarrow{\Delta} \mathcal{F}(S) \times \mathcal{F}(S)$.

Theorem (Bonahon '88)

Let g_{WP} be the Weil-Petersson metric on $\mathcal{F}(S)$, and suppose $m(t,u) \in \mathcal{F}(S)$ is any differentiable 2-parameter family (set $m = m(0,0)$). Then

$$g_{WP}\left(\frac{\partial}{\partial t} m(t,u), \frac{\partial}{\partial u} m(t,u)\right)_m = \frac{3}{\pi} \cdot \frac{\partial^2}{\partial t \partial u} i(m, m(t,u))$$

Reparameterizations of Geodesic Flow

$$\{\text{closed geodesics in } S\} \Longleftrightarrow \{\text{closed orbits for the geodesic flow on } T^1S\}$$

A choice of new hyperbolic structure on S determines a reparameterization of the geodesic flow on T^1S. For $\rho \in \mathcal{F}(S)$, $\gamma \in \pi_1(S)$, we have

$$\begin{aligned}
\ell_\rho(\gamma) &= \text{length of closed geodesic in } [\gamma] \text{ in the metric determined by } \rho \\
&= \text{period of a closed orbit corresponding to } [\gamma] \text{ under geodesic flow} \\
&= \text{translation length of } \rho(\gamma) \text{ in } \mathbb{H}^2 = 2\log(\Lambda(\rho(\gamma)))
\end{aligned}$$

Use this to define reparameterizations according to data from $\rho \in \mathcal{H}_d(S)$?

Reparameterization from a Hitchin Representation

Theorem (Sambarino '14)

Given $\rho \in \mathcal{H}_d(S)$, there exists a Hölder $f_\rho : T^1S \to \mathbb{R}^+$ such that $\forall \gamma \in \pi_1(S)$,

$$\ell_{f_\rho}(\gamma) = \int_{[\gamma]} f_\rho = \log(\Lambda(\rho(\gamma))).$$

If $\phi : X \times \mathbb{R} \to X$ a smooth flow, $f : X \to \mathbb{R}^+$ Hölder,

ϕ_f is the reparameterization of ϕ with speed scaled by the function f

Markov Codings

A **Markov coding** for a flow encodes flow data into a symbolic system

Intuitively: chop the space up into pieces for which the dynamics of the flow resemble that of a subshift of finite type

Theorem (Bowen '73, Pollicott '87)

Topologically transitive (metric) Anosov flows admit Markov codings (hence geodesic flows on compact manifolds of negative curvature admit codings).

Yields access to the thermodynamic formalism for subshifts of finite type

Thermodynamic Formalism

Suppose $\phi : X \times \mathbb{R} \to X$ is a topologically transitive, metric Anosov flow.
Let $R_T(\phi) = \{\gamma \text{ a closed orbit for } \phi \text{ with period } p(\gamma) \leq T\}$ and $\ell_f(\gamma) = \int_{[\gamma]} f$.

Corollary (of the existence of Markov codings)

The topological entropy $h(\phi)$ is given by

$$h(\phi) = \lim_{T\to\infty} \frac{1}{T} \log\left(\#R_T(\phi)\right).$$

The pressure $\mathbf{P}(f)$ of a Hölder continuous $f : X \to \mathbb{R}$ is given by

$$\mathbf{P}(f) = \mathbf{P}(f,\phi) = \lim_{T\to\infty} \frac{1}{T} \log\left(\sum_{\gamma\in R_T(\phi)} \exp\left(\ell_f(\gamma)\right)\right) = \sup_{\mu\in\mathcal{M}^\phi}\left(h_\mu(\phi) + \int_X f\, d\mu\right),$$

where h_μ is the measure-theoretic entropy.

The Pressure Norm for Functions

Definition (Livšic)

Suppose $\phi : X \times \mathbb{R} \to X$ is a flow and $f, g : X \to \mathbb{R}$. Then f and g are **Livšic cohomologous** (denoted $f \sim g$) if there is $V : X \to \mathbb{R}$ which is C^1 along orbits, so that $f(x) - g(x) = \frac{d}{dt}|_{t=0} V(\phi_t(x))$.

Theorem (McMullen, Bridgeman-Canary-Labourie-Sambarino)

Let $\mathcal{P}(X) = \{f : X \to \mathbb{R} \mid \mathbf{P}(f) = 0\}/\sim$. *Then for* $f \in \mathcal{P}(X)$, $T_f\mathcal{P}(X) = \ker d_f\mathbf{P} = \{g : X \to \mathbb{R} \mid \int g \, dm_f = 0\}/\sim$, *and the formula*

$$\|g\|_f^2 = \frac{1}{-\int f \, dm_f} \cdot \frac{d^2}{dt^2}\bigg|_{t=0} \mathbf{P}(f + tg)$$

defines a norm on $T_f\mathcal{P}(X)$.

The measures m_f in the above theorem are called **equilibrium states** and in our context they are unique measures associated to a function $f : X \to \mathbb{R}$

Connection to Weil-Petersson

McMullen's context: pressure 0 functions on discrete time symbolic systems.

Theorem (McMullen '06)

The pressure metric pulls back to a multiple of the Weil-Petersson metric under a thermodynamic embedding $\mathcal{T}(S) \hookrightarrow \mathcal{P}(\Sigma)$, determined by the Bowen-Series coding.

Bridgeman '10: constructed the pressure metric for quasifuchsian space

Hitchin Representations to Hölder Functions

Pressure norm for functions $\Longrightarrow$ to Hitchin reps?

How to associate a pressure 0 function to an Hitchin representation? Answer:

Theorem (Sambarino '14)

If ϕ is a topologically transitive Anosov flow on X and $f : X \to \mathbb{R}^+$ is Hölder,

$$\mathbf{P}(-hf) = 0 \Longleftrightarrow h = h(\phi_f).$$

The Thermodynamic Mapping

We then define the **thermodynamic mapping**,

$$\Phi : \mathcal{H}_d(S) \to \mathcal{P}(T^1 S)/\sim$$

by

$$\rho \mapsto [-h_{f_\rho} f_\rho],$$

where f_ρ is the reparameterization function associated to the representation ρ.

Pull back under Φ to obtain the pressure metric for Hitchin representations.

Intersection for Hölder Reparameterizations

Motivated by Bonahon:

Definition (BCLS '15)

Suppose $f_1, f_2 : X \to \mathbb{R}^+$ are Hölder. Then define the **intersection form** I by

$$I(f_1, f_2) = \lim_{T\to\infty} \left(\frac{1}{\#R_T(f_1)} \sum_{\gamma \in R_T(f)} \frac{\ell_{f_2}(\gamma)}{\ell_{f_1}(\gamma)} \right)$$

and **renormalized intersection form** $J(\cdot, \cdot)$ by

$$J(f_1, f_2) = \frac{h(\phi_{f_1})}{h(\phi_{f_2})} I(f_1, f_2)$$

The Pressure Metric for Hitchin Representations

Given representations $\rho_1, \rho_2 \in \mathcal{H}_d(S)$, let $f_1, f_2 : T^1S \to \mathbb{R}^+$ be the corresponding reparameterization functions for the geodesic flow on T^1S. Define

$$I(\rho_1, \rho_2) = I(f_1, f_2) \text{ and } J(\rho_1, \rho_2) = J(f_1, f_2).$$

Theorem (BCLS)

If $\rho_t \in \mathcal{H}_d(S)$ is a smooth one-parameter family with reparameterization functions $f_t : T^1S \to \mathbb{R}^+$, then set $\Phi_t = \Phi(\rho_t) = -h(\phi_{f_t})f_t$, and we have

$$\|\dot{\Phi}_0\|_{\mathbf{P}}^2 = \frac{\partial^2}{\partial t^2}\bigg|_{t=0} J(f_0, f_t).$$

Thus, for $\rho \in \mathcal{H}_d(S)$, we get the pressure form p, a symmetric 2-tensor on $T_\rho\mathcal{H}_d(S)$, as the Hessian of $J_\rho = J(\rho, \cdot)$

Open Questions

- Geometry of Hitchin components? Finite or infinite diameter?
- Geometry of the Fuchsian locus inside the Hitchin component?
- Curvature of the pressure metric?
- Interpretation of the completion?
- Topological structure of the space of Anosov representations?
- Behavior of entropy for Hitchin representations?

Hitchin Grafting Representations

From Blayac-Hamenstädt-Marty-Monti ’24:

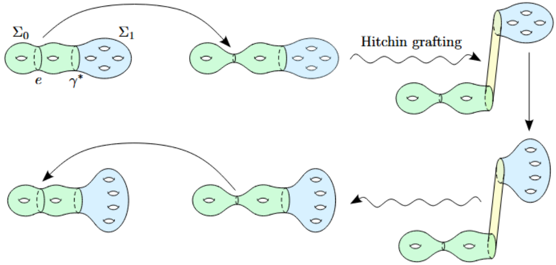


Figure 6: Bounded path of Hitchin representations for the pressure metric. Each path is bounded by a constant that depends only on the length of γ^* and on the systole of Σ_0.

Produces subsets of infinite diameter under Weil-Petersson in the Fuchsian locus, but finite diameter in the path metric of the pressure metric

Possible Test Cases

Low dimensional examples?

Triangle reflection groups: 1-dimensional Hitchin components

$\mathcal{H}_3(S)$: use associated geometric structures?

Internal sequences: entropy $\to 0$, distance from the Fuchsian locus?

Primary References

Bonahon (1988)
Geodesic currents and Teichmüller space

Blayac, Hamenstädt, Marty, Monti (2024)
Geometry and dynamics of Hitchin grafting representations

Bridgeman, Canary, Labourie, Sambarino (2015)
The Pressure Metric for Anosov Representations

Canary (2021)
Anosov Representations: Informal Lecture Notes

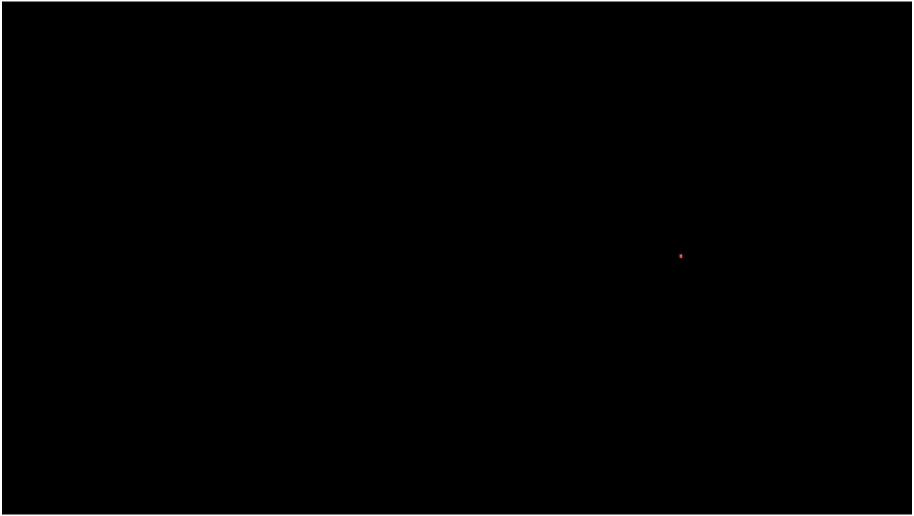
Hitchin (1992)
Lie groups and Teichmüller space

Kassel (2024)
Discrete subgroups of semisimple Lie groups, beyond lattices

Labourie (2006)
Anosov Flows, Surface Groups and Curves in Projective Space

McMullen (2006)
Thermodynamics, dimension, and the Weil-Petersson metric

The End



Bonus 1: Geodesic Flow of a Hyperbolic Group

Theorem (Gromov, Champetier, Mineyev)

Let Γ be a hyperbolic group, $U(\Gamma) = (\partial\Gamma \times \partial\Gamma \setminus \Delta) \times \mathbb{R}$. There exists a complete metric on $U(\Gamma)$, a flow $\phi_t : U(\Gamma) \to U(\Gamma)$, and a properly discontinuous, cocompact action of $\Gamma \curvearrowright U(\Gamma)$ by isometries for which

- *The map $t \mapsto (z, w, t) \in U(\Gamma)$ is a quasi-isometric embedding;*
- *$\phi_t(z, w, s) = (z, w, s + t)$, and each ϕ_t is bi-Lipschitz;*
- *The action of Γ commutes with ϕ_t, and the flow descends to a topologically transitive flow $\hat{\phi}_t$ on $\hat{U}(\Gamma) = U(\Gamma)/\Gamma$;*
- *For $\gamma \in \Gamma$, there is a function $c_\gamma : \partial\Gamma \times \partial\Gamma \setminus \Delta \to \mathbb{R}$ such that $\gamma(x, y, t) = (\gamma x, \gamma y, c_\gamma(x, y) + t)$.*

If $\Gamma = \pi_1(S)$, we have a flow-equivariant homeomorphism $\hat{U}(\Gamma) \cong T^1 S$

Bonus 2: The Right Way to define Anosov Representations

Definition

A representation $\rho : \Gamma \to \mathbf{SL}_m(\mathbb{R})$ has **transverse projective limit maps** if there exist ρ-equivariant continuous maps $\xi : \partial\Gamma \to \mathbb{RP}(m)$ and $\eth : \partial\Gamma \to \mathbb{RP}(m)^*$ such that if $x \neq y \in \partial_\infty\Gamma$, then

$$\xi(x) \oplus \eth(y) = \mathbb{R}^m$$

A representation ρ with transverse projective limit maps $\xi, \eth$ is **projective Anosov** if the induced splitting of the flat bundle

$$E_\rho = (U(\Gamma) \times \mathbb{R}^m)/\Gamma = \Xi \oplus \Theta$$

(which is parallel along the geodesic flow) satisfies that the bundle $T_\Theta \mathbb{RP}(m)^* \cong \mathrm{Hom}(\Theta, \Xi) \cong \Xi \otimes \Theta^*$ is contracted by the flow (for any metric).

Bonus 3: Geodesic Flow of an Anosov Representation

Fix $\rho : \Gamma \to \mathbf{SL}_m(\mathbb{R})$ a projective Anosov representation with limit maps ξ, ∂

Definition (Geodesic flow of a projective Anosov representation)

Let F be the total space of the bundle over

$$B = \mathbb{RP}(m) \times \mathbb{RP}(m)^* \setminus \{(U, V) \mid U \not\subseteq V\}$$

with fiber at (U, V) given by

$$M(U, V) = \{(u, v) \mid u \in U, v \in V, \langle v \mid u \rangle = 1\}/(u, v) \sim (-u, -v).$$

F is an $\mathbb{R}$-bundle, and there is a flow by $\phi_t(U, V, (u, v)) = (U, V, (e^t u, e^{-t} v))$.
Set

$$F_\rho = (\xi, \partial)^* F \;\text{ and }\; U_\rho(\Gamma) = F_\rho / \Gamma.$$

The flow is inherited, and the flow on $U_\rho(\Gamma)$ is called the geodesic flow of ρ.

Bonus 4: Geodesic Currents

Theorem (Bonahon '88)

- *A hyperbolic metric m on S determines a current L_m (called the* **Liouville current** *of m) and $m \mapsto L_m$ is a proper embedding $\mathcal{F}(S) \hookrightarrow \mathcal{C}(S)$;*
- *A measured geodesic lamination determines an element of $\mathcal{C}(S)$, so there is a (dense) embedding $\mathbb{R}^+[\pi_1(S)] \hookrightarrow \mathcal{C}(S)$;*
- *There is a continuous symmetric, bilinear* **intersection form** *$i : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}^{\geq 0}$, where $i(\alpha, \beta)$ is the total mass of $\alpha \times \beta$ on the space of couples of geodesics with transverse intersection.*

Theorem (Properties of i)

- *When $\alpha, \beta \in \mathcal{C}(S)$ are distinct closed geodesics on S, $i(\alpha, \beta) = |\alpha \cap \beta|$;*
- *For $m \in \mathcal{F}(S)$, $i(L_m, L_m) = \pi^2|\chi(S)|$ (note no dependence on m);*
- *If $m, m' \in \mathcal{F}(S)$ are distinct, then $i(L_m, L'_m) \geq \pi^2|\chi(S)|$;*
- *If $m \in \mathcal{F}(S)$ and α is a closed curve on S, then $i(\alpha, L_m) = \ell_m([\alpha])$.*

Bonus 5: Entropy and Pressure (Abstractly)

Let $T : X \to X$ be a continuous transformation on a compact metric space, and define the sequence d_n of metrics on X by $d_n(x, y) = \max_{0 \le i \le n-1} d(T^i x, T^i y)$.

Definition (Topological Entropy)

A subset $B \subseteq X$ is said to be (n, ε)-**separated** if for any two $b, b' \in B$, $b \neq b'$ implies $d_n(b, b') > \varepsilon$. Let $\operatorname{sep}(n, \varepsilon, T) = \max \#$ of an (n, ε)-separated set in X. Then the **topological entropy** of T, denoted $h(T)$, is given by

$$h(T) = \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log(\operatorname{sep}(n, \varepsilon, T)) \right).$$

The simplest characterization for pressure is

Theorem (Variational Principle)

$$\mathbf{P}(f) = \mathbf{P}(f, T) = \sup_{\mu \in \mathcal{M}_T} \left(h_\mu(T) + \int_X f \, d\mu \right)$$

Bonus 6: Details for Markov Codings

A **Markov coding** for a flow $\phi : X \times \mathbb{R} \to X$ is a triple (Σ, r, π) where

- Σ is an irreducible (two-sided) subshift of finite type, with shift map σ
- $r : \Sigma \to \mathbb{R}^+$ is a Hölder function, so we may consider the flow over Σ under r (the vertical flow on $\Sigma_r = \{(x, t) \in \Sigma \times \mathbb{R} \mid 0 \leq t \leq r(x)\}/\sim$)
- $\pi : \Sigma_r \to X$ is a topological semi-conjugacy of flows, bounded-to-one and with measure 0 failure of injectivity

Straightforward exercise: reparameterization of a flow admitting a Markov coding maintains this property

Bonus 7: Details on Teichmüller and Fricke Space

$$\mathcal{T}(S) = \{(X, f) \mid X \text{ a Riemann surface}, f : S \to X \text{ a holo. diffeo.}\}/\text{Isotopy}$$
$$\mathrm{Hyp}(S) = \{g \mid g \text{ a Riemannian metric on } S \text{ of constant curvature } -1\}/\mathrm{Diff}_0(S)$$
$$\mathcal{F}(S) = \{\rho : \pi_1(S) \to \mathbf{PSL}_2(\mathbb{R}) \mid \rho \text{ discrete and faithful}\}/\mathbf{PGL}_2(\mathbb{R})$$

Bonus 8: Details on Rigidity Phenomena

Theorem (Local Rigidity)

Suppose G semisimple with no simple factors that are compact or locally isomorphic to $\mathbf{PSL}_2(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). If Γ is a cocompact irreducible lattice of G, then any continuous path $\rho_t : \Gamma \to G$ is of the form $\rho_t = g_t \rho_0(\cdot) g_t^{-1}$.

Theorem (Mostow Rigidity)

Let $n \geq 3$. If $\Lambda, \Gamma \leqslant \mathbf{PO}(n,1) = Isom^+(\mathbb{H}^n)$ are lattices and $\Lambda \cong \Gamma$, then in fact $\exists g \in \mathbf{PO}(n,1)$ so that $g\Lambda g^{-1} = \Gamma$.

Theorem (Corollary of Margulis Superrigidity)

Let $k \geq 3$. If $\Gamma \leqslant \mathbf{SL}_k(\mathbb{R})$ such that $\mathbf{SL}_k(\mathbb{R})/\Gamma$ is not compact, and $\phi : \Gamma \to \mathbf{GL}_n(\mathbb{R})$ is any homomorphism. Then there is a finite index $\Gamma' \leqslant \Gamma$ and $\hat{\phi} : \mathbf{SL}_k(\mathbb{R}) \to \mathbf{GL}_n(\mathbb{R})$ such that $\hat{\phi}|_{\Gamma'} \equiv \phi|_{\Gamma'}$.

Bonus 9: More pictures from BHMM '24

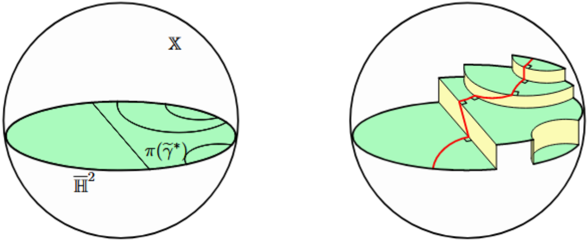


Figure 1: Geometric description of the Hitchin representation: the hyperbolic part in green, the flat part in yellow and an admissible path in red.

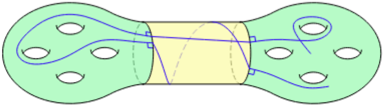


Figure 2: Admissible path in the closed surface obtained as an abstract grafting.