A Topological Proof of The Fundamental Theorem of Algebra

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Theorem 1 (The Fundamental Theorem of Algebra)

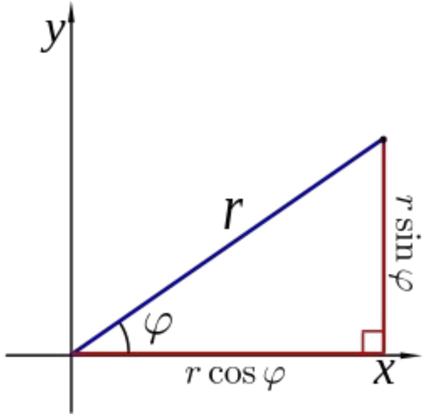
Let $f(z)$ be a non-constant complex polynomial.
Then $f(z)$ has a root in $\mathbb{C}$.

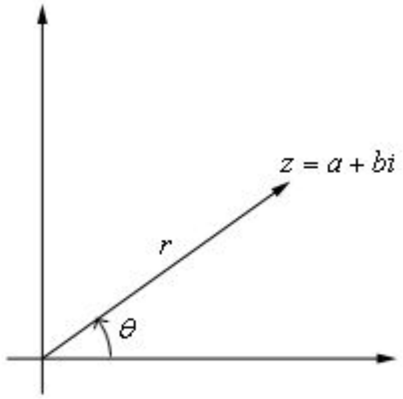
The Strategy of the Proof

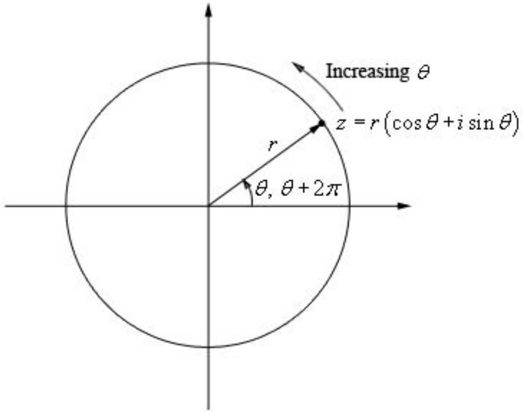
- We will be proving the contrapositive
- Let f(z) be a complex polynomial
- If f(z) has no complex roots, then f(z) must be constant
- We want to consider what the function f(z) does to the circle
- The reason for this is that the circle is topologically well-understood

Polar Representation of Complex Numbers

- Any complex number z can be written in terms of variables r and θ
- What most people are familiar with: $z = a + bi$
- What we will use here: $z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$







The Proof

Let $p(z)$ be an arbitrary complex polynomial of degree n.
We may assume $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$.
Suppose $p(z)$ has no roots in $\mathbb{C}$.

Which Paths do We Want to Consider?

We want to consider how our polynomial acts on the circle. $re^{2\pi is}$ as $s : 0 \to 1$ is a parameterization of the circle of radius r. We should consider the path given by $p(re^{2\pi is})$ as $s : 0 \to 1$.

Let’s Fix the Path

We would also like to ensure that the angle of our basepoint is at $\theta = 0$.
To accomplish this, we divide by the constant $p(r)$.
So now our path is given by $p(re^{2\pi is})/p(r)$.

Let's Fix the Path... Again

Finally, we would like for our paths to exist on the unit circle. To ensure this, we simply divide by the modulus at any point. This yields the path

$$\frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

Here is the Desired Formula

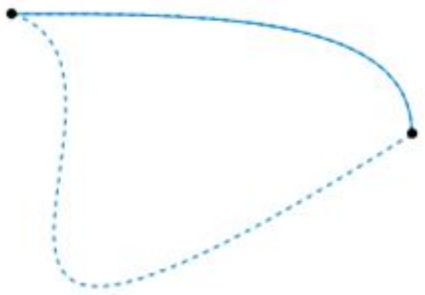
Thus, for every real number r, we define the formula $f_r : I \to \mathbb{C}$ by

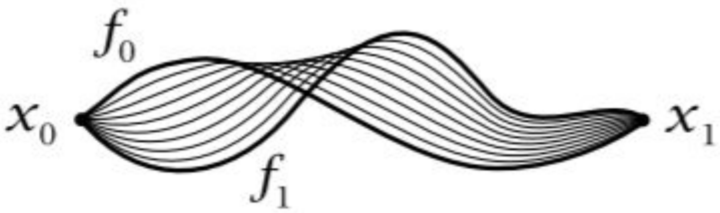
$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

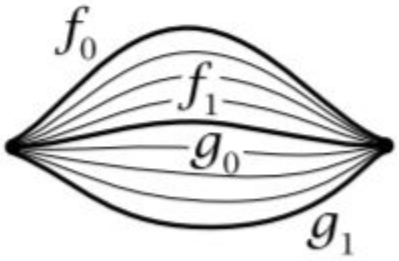
This defines a loop in the unit circle $S^1 \subset \mathbb{C}$ based at the point $z_0 = 1$.

Homotopy of paths

- Two paths are said to be “homotopic” if there is a continuous mapping from one to the other
- This defines an equivalence relation on the set of loops at a point
- The property we will use: transitivity







This Formula Gives A Homotopy

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

Making use of the fact that $p(z) \neq 0$, we see that as r varies, f_r is a homotopy of loops based at $z_0 = 1$.

Why is This Significant?

Note that f_0 is the trivial loop, as $f_0(s) = \frac{p(0)/p(0)}{|p(0)/p(0)|} \equiv 1$ for any s.
Thus for any r, the loop f_r is homotopic to a constant loop (it is said to be nullhomotopic).

Let’s Build Another Homotopy

Now, fix a value of r which is larger than $|a_{n-1}| + \ldots + |a_0|$ and larger than 1. Then on the circle $|z| = r$, we see that

$$|z^n| > (|a_{n-1}| + \ldots + |a_0|)|z^{n-1}| > |a_{n-1}z^{n-1}| + \ldots + |a_0| \geq |a_{n-1}z^{n-1} + \ldots + a_0|$$

Another Homotopy

From the inequality $|z^n| > |a_{n-1}z^{n-1} + ... + a_0|$, it follows that the polynomial $p_t(z) = z^n + t(a_{n-1}z^{n-1} + ... + a_0)$ has no roots on the circle $|z| = r$ when $t \in [0, 1]$. Also note $p_1(z) = p(z)$ and $p_0(z) = z^n$.

https://www.desmos.com/calculator/trypikvlji

The Induced Homotopy of Their Paths

Replacing p by p_t in the formula for f_r and letting t run from 1 to 0 gives us a homotopy between f_r and the loop $\omega_n(s) = e^{2\pi ins}$.

$$\frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|}$$

When $t = 1$,

$$\frac{p_1(re^{2\pi is})/p_1(r)}{|p_1(re^{2\pi is})/p_1(r)|} = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|} = f_r(s)$$

When $t = 0$,

$$\frac{p_0(re^{2\pi is})/p_0(r)}{|p_0(re^{2\pi is})/p_0(r)|} = \frac{(re^{2\pi is})^n/(r)^n}{|(re^{2\pi is})^n/(r)^n|} = r^n e^{2\pi ins}/r^n = e^{2\pi ins} = \omega_n(s)$$

Why does Omega matter?

Note that ω_n is precisely the loop that wraps around the circle n times.

The Fundamental Group

- Algebraic structure associated with loops in the space
- For the circle, we associate loops with the number of times they wrap around the circle
- Two loops are associated with the same integer if and only if the loops are homotopic

The Punchline

$$n \iff \omega_n \simeq f_r \simeq f_0 \iff 0$$

$$\Downarrow$$

$$n = 0$$